Paper

Decision Support under Risk by Optimization of Scenario Importance Weighted OWA Aggregations

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Abstract—The problem of evaluation outcomes under several scenarios to form overall objective functions is of considerable importance in decision support under uncertainty. The fuzzy operator defined as the so-called weighted OWA (WOWA) aggregation offers a well-suited approach to this problem. The WOWA aggregation, similar to the classical ordered weighted averaging (OWA), uses the preferential weights assigned to the ordered values (i.e., to the worst value, the second worst and so on) rather than to the specific criteria. This allows one to model various preferences with respect to the risk. Simultaneously, importance weighting of scenarios can be introduced. In this paper we analyze solution procedures for optimization problems with the WOWA objective functions related to decisions under risk. Linear programming formulations are introduced for optimization of the WOWA objective representing risk averse preferences. Their computational efficiency is demonstrated.

***Keywords—* aggregation methods, decisions under risk, OWA, scenarios, WOWA.**

1. Introduction

In decision problems under uncertainty, we consider, the decision is based on the maximization of a scalar (real valued) outcome. The final outcome is uncertain and only its realizations under various scenarios are known. Exactly, for each scenario S_i ($i \in I = \{1, 2, \ldots, m\}$) the corresponding outcome realization is given as a function of the decision variables $y_i = f_i(\mathbf{x})$. We are interested in larger outcomes under each scenario. Hence, the decision under uncertainty can be considered a multiple criteria optimization problem:

$$\max \; \{ \; (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \; : \; \mathbf{x} \in \mathscr{F} \; \}, \qquad (1)$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $\mathscr{F} \subset R^q$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set $\mathscr{F}$ into the criterion space R^m. The feasible set $\mathscr{F}$ is usually defined by some constraints. The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in \mathscr{F}$ ($\mathbf{y} = \mathbf{f}(\mathbf{x})$). The set of all the attainable achievement vectors is denoted by A, i.e., $A = \{\mathbf{y} = \mathbf{f}(\mathbf{x}) \; : \; \text{for some } \mathbf{x} \in \mathscr{F}\}$.

From the perspective of decisions under uncertainty, model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts are defined by aggregation functions $a : R^m \to R$. Thus the multiple criteria problem (1) is replaced with the (scalar) maximization problem:

$$\max \; \{a(\mathbf{f}(\mathbf{x})) \; : \; \mathbf{x} \in \mathscr{F}\}.$$

The most commonly used aggregation is based on the weighted mean where positive importance weights p_i ($i = 1, \ldots, m$) are allocated to several scenarios:

$$A_{\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} y_i p_i \,. \qquad (2)$$

The weights are typically normalized to the total 1 ($\sum_{i=1}^{m} p_i = 1$) with possible interpretation as scenarios (subjective) probabilities. The weighted mean enables to define the importance of scenarios but it does not allow one to model the decision maker's preferences regarding the distribution of outcomes. The latter is crucial when aggregating various realizations of the same (uncertain) outcome under several scenarios and one needs to model risk averse preferences [1].

The preference weights can be effectively introduced within the fuzzy optimization methodology with the so-called ordered weighted averaging (OWA) aggregation [2]. In the OWA aggregation the weights are assigned to the ordered values (i.e., to the largest value, the second largest and so on) rather than to the specific criteria. This guarantees a possibility to model various preferences with respect to the risk. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [3]–[6]. The weighting of the ordered outcome values causes that the OWA optimization problem is nonlinear even for linear programming (LP) formulation of the original constraints and criteria. Yager [7] has shown that the OWA optimization can be converted into a mixed integer programming problem. We have shown [8], [9] that the OWA optimization with monotonic weights can be formed as a standard linear program of higher dimension.

The OWA operator allows one to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow one to allocate any importance weights to specific scenarios. Actually, the weighted mean (2) cannot be expressed in terms of the OWA aggregations. Several attempts have been made to incorporate importance weighting into the OWA operator [10], [11]. Finally, Torra [12] has incorporated importance weighting into the OWA operator within the weighted OWA (WOWA) aggregation introduced as a particular case of Choquet integral using a distorted probability as the measure. The WOWA averaging is defined by two weighting vectors: the preferential weights **w** and the importance weights **p**. It covers both the weighted means (defined with **p**) and the OWA averages (defined with **w**) as special cases. Actually, the WOWA average becomes the weighted mean in the case of equal all the preference weights and it is reduced to the standard OWA average for equal all the importance weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [13], [14] including metadata aggregation problems [15], [16].

In this paper we analyze solution procedures for optimization problems with the WOWA objective functions modeling decisions under risk. A linear programming formulations are introduced for optimization of the WOWA objective with increasing preferential weights thus representing risk averse preferences. The paper is organized as follows. In Section 2 we introduce formally the WOWA operator and derive some alternative computational formula based on direct application of the preferential weights to the conditional means according to the importance weights. Further, in Section 3, we analyze the orness/andness properties of the WOWA operator with monotonic preferential weights and the corresponding risk profiles. In Section 4 we introduce the LP formulations for maximization of the WOWA aggregation with increasing weights. Finally, in Section 5 we demonstrate computational efficiency of the introduced models.

2. The WOWA Aggregation

Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m w_i = 1$. The corresponding OWA aggregation of outcomes $\mathbf{y} = (y_1, \ldots, y_m)$ can be mathematically formalized as follows [2]. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e., the OWA aggregation is defined as follows:

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}), \tag{3}$$

where $w_i \geq 0$ for $i = 1, \ldots, m$ are normalized weights ($\sum_{i=1}^m w_i = 1$). The OWA aggregation (3) allows one to model various aggregation functions from the maximum ($w_1 = 1$, $w_i = 0$ for $i = 2, \ldots, m$) through the arithmetic mean ($w_i = 1/m$ for $i = 1, \ldots, m$) to the minimum ($w_m = 1$, $w_i = 0$ for $i = 1, \ldots, m-1$).

Now, let again $\mathbf{w} = (w_1, \ldots, w_m)$ be an m-dimensional vector of preferential weights $w_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m w_i = 1$. Additionally, let $\mathbf{p} = (p_1, \ldots, p_m)$ be an m-dimensional vector of importance weights such that $p_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m p_i = 1$. The corresponding weighted OWA aggregation of vector $\mathbf{y} = (y_1, \ldots, y_m)$ is defined [12] as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}) \tag{4}$$

with the weights ω_i defined as

$$\omega_i = w^*\Big(\sum_{k \leq i} p_{\tau(k)}\Big) - w^*\Big(\sum_{k < i} p_{\tau(k)}\Big), \tag{5}$$

where w^* is an increasing function interpolating points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e., $y_{\tau(i)} = \theta_i(\mathbf{y})$). Moreover, function w^* is required to be a straight line when the point can be interpolated in this way. For our purpose of decision support under risk we will focus on the linear interpolation thus leading to the piecewise function w^*.

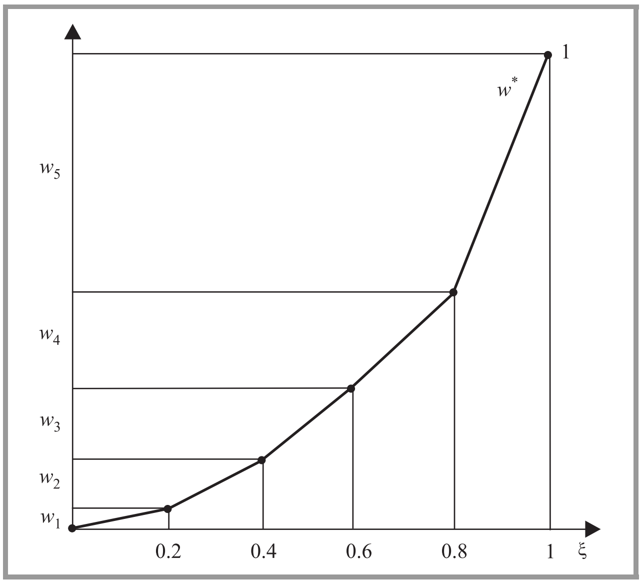

Fig. 1. Function w^* for $\mathbf{w} = (0.05, 0.1, 0.15, 0.2, 0.5)$.

To illustrate the WOWA average let us consider two outcome vectors $\mathbf{y}' = (3,1,2,4,5)$ and $\mathbf{y}'' = (1,1,2,6,4)$, where individual outcomes correspond to five scenarios. While introducing preferential weights $\mathbf{w} = (0.05, 0.1, 0.15, 0.2, 0.5)$ one may calculate the OWA averages: $A_{\mathbf{w}}(\mathbf{y}') = 0.05 \cdot 5 + 0.1 \cdot 4 + 0.15 \cdot 3 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2$ and $A_{\mathbf{w}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.1 \cdot 4 + 0.15 \cdot 2 + 0.2 \cdot 1 + 0.5 \cdot 1 = 1.7$. Further, let us introduce importance weights $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$ which means that results under the third scenario are 2 times more important then those under scenario 1, 2 or 5, while the results under scenario 4 are even 5 times more important. To take into account the importance weights in the WOWA aggregation (4) we introduce the following piecewise linear function (cf. Fig. 1):

$$w^*(\xi) = \begin{cases} 0.05\xi/0.2, & 0 \le \xi \le 0.2 \\ 0.05 + 0.10(\xi - 0.2)/0.2, & 0.2 < \xi \le 0.4 \\ 0.15 + 0.15(\xi - 0.4)/0.2, & 0.4 < \xi \le 0.6 \\ 0.3 + 0.2(\xi - 0.6)/0.2, & 0.6 < \xi \le 0.8 \\ 0.5 + 0.5(\xi - 0.8)/0.2, & 0.8 < \xi \le 1.0 \end{cases}$$

and calculate weights ω_i according to formula (4) as w^* increments corresponding to importance weights of the ordered outcomes, as illustrated in Fig. 2. In particular, one get $\omega_1 = w^*(p_5) = 0.025$ and $\omega_2 = w^*(p_5 + p_4) - w^*(p_5) = 0.275$ for vector $\mathbf{y}'$ while $\omega_1 = w^*(p_4) = 0.225$ and $\omega_2 = w^*(p_4 + p_5) - w^*(p_4) = 0.075$ for vector $\mathbf{y}''$. Finally,

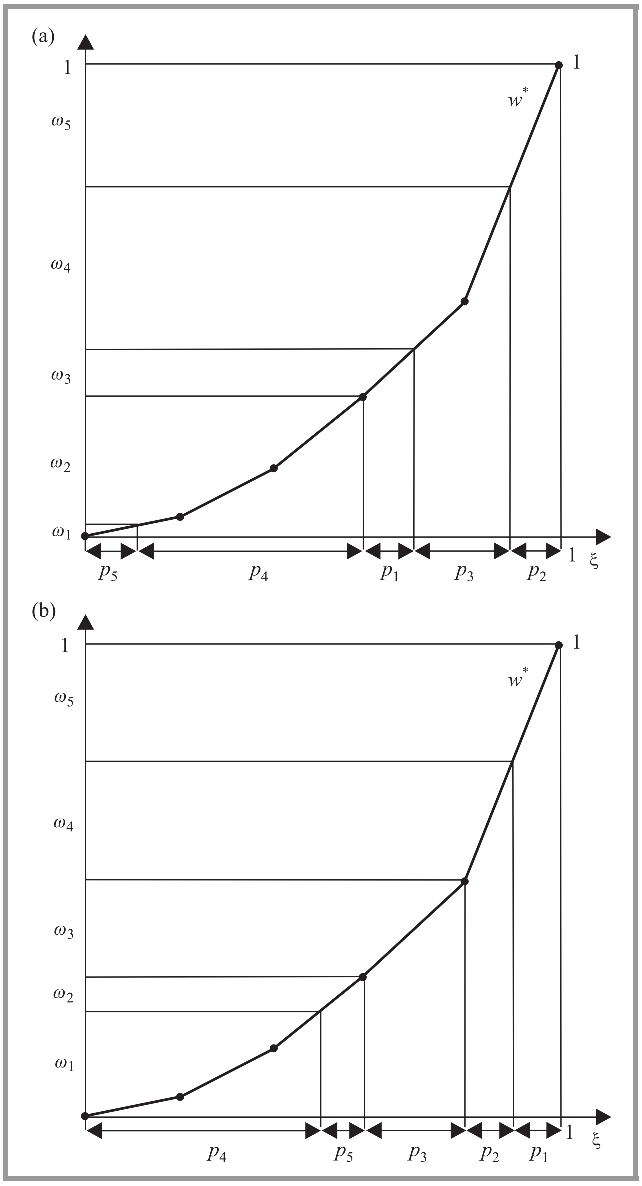

Fig. 2. Definition of weights ω_i for WOWA formula (4) for $\mathbf{w} = (0.05, 0.1, 0.15, 0.2, 0.5)$ and $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$: (a) vector $\mathbf{y}' = (3, 1, 2, 4, 5)$; (b) vector $\mathbf{y}'' = (1, 1, 2, 6, 4)$.

$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.025 \cdot 5 + 0.275 \cdot 4 + 0.1 \cdot 3 + 0.35 \cdot 2 + 0.25 \cdot 1 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.225 \cdot 6 + 0.075 \cdot 4 + 0.2 \cdot 2 + 0.25 \cdot 1 + 0.25 \cdot 1 = 2.55$.

Note that one may alternatively compute the WOWA values by using the importance weights to replicate corresponding scenarios and calculate then OWA aggregations. In the case of our importance weights $\mathbf{p}$ we need to consider five copies of scenario 4 and two copies of scenario 3 thus generating corresponding vectors $\tilde{\mathbf{y}}' = (3, 1, 2, 2, 4, 4, 4, 4, 4, 5)$ and $\tilde{\mathbf{y}}'' = (1, 1, 2, 2, 6, 6, 6, 6, 6, 4)$ of ten equally important outcomes. Original five preferential weights must be then applied respectively to the average of the two largest outcomes, the average of the next two largest outcomes, etc. Indeed, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.05 \cdot 4.5 + 0.1 \cdot 4 + 0.15 \cdot 4 + 0.2 \cdot 2.5 + 0.5 \cdot 1.5 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.1 \cdot 6 + 0.15 \cdot 5 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2.55$. We will further formalize this approach and take its advantages to build the LP computational models.

Function w^* can be defined by its generation function g with the formula $w^*(\alpha) = \int_0^\alpha g(\xi)\, d\xi$. Introducing breakpoints $\alpha_i = \sum_{k \le i} p_{\tau(k)}$ and $\alpha_0 = 0$ we get

$$\omega_i = \int_0^{\alpha_i} g(\xi)\, d\xi - \int_0^{\alpha_{i-1}} g(\xi)\, d\xi = \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\, d\xi$$

and finally [17], [18]:

$$\begin{aligned} A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) &= \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\alpha_{i-1}}^{\alpha_i} g(\xi)\, d\xi \\ &= \int_0^1 g(\xi) \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi\,, \end{aligned} \tag{6}$$

where $\overline{F}_{\mathbf{y}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\alpha_{i-1} < \xi \le \alpha_i$. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i=1}^{m} p_i \delta_i(d)\,, \tag{7}$$

where

$$\delta_i(d) = \begin{cases} 1 & \text{if } y_i \le d \\ 0 & \text{otherwise} \end{cases}$$

which for any real (outcome) value d provides the measure of outcomes smaller or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)} = \inf\{\eta : F_{\mathbf{y}}(\eta) \ge \xi\}$ for $0 < \xi \le 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$, and finally $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = F_{\mathbf{y}}^{(-1)}(1 - \xi)$.

Formula (6) provides the most general expression of the WOWA aggregation allowing for expansion to continuous case. The original definition of WOWA allows one to build various interpolation functions w^* [19] thus to use different generation functions g in formula (6). Let us focus our analysis on the the piecewise linear interpolation function w^*. It is the simplest form of the interpolation function. Note, however, that the piecewise linear functions may be built with various number of breakpoints, not necessarily m. Thus, any nonlinear function can be well approximated by a piecewise linear function with appropriate number of breakpoints. Therefore, we will consider weights vectors $\mathbf{w}$ of dimension n not necessarily equal to m. Any such piecewise linear interpolation function w^* can be expressed with the stepwise generation function:

$$g(\xi) = nw_k \text{ for } (k-1)/n < \xi \leq k/n, \quad k = 1,\ldots,n. \quad (8)$$

This leads us to the following specification of formula (6):

$$\begin{aligned} A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) &= \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi \\ &= \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi\,. \end{aligned} \quad (9)$$

Note that $n\int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ represents the average within the kth portion of $1/n$ largest outcomes, the corresponding conditional mean [20], [21]. Hence, formula (9) defines WOWA aggregations with preferential weights $\mathbf{w}$ as the corresponding OWA aggregation but applied to the conditional means calculated according to the importance weights $\mathbf{p}$ instead of the original outcomes. Figure 3 illustrates application of formula (9) to computation of the WOWA aggregations for vectors from Fig. 2.

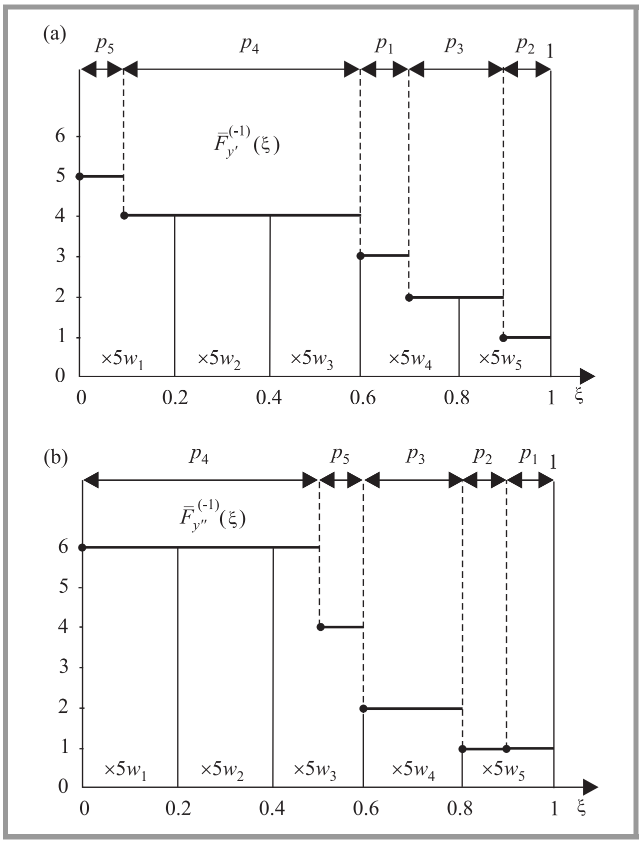

Fig. 3. Formula (9) applied to WOWA calculations for $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$: (a) vector $\mathbf{y}' = (3,1,2,4,5)$; (b) vector $\mathbf{y}'' = (1,1,2,6,4)$.

We will treat formula (9) as a formal definition of the WOWA aggregation of m-dimensional outcomes $\mathbf{y}$ defined by the m-dimensional importance weights $\mathbf{p}$ and the n-dimensional preferential weights $\mathbf{w}$. Formula (9) may be reformulated to use the tail averages:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} nw_k \Big(L\big(\mathbf{y},\mathbf{p},1-\frac{k-1}{n}\big) - L\big(\mathbf{y},\mathbf{p},1-\frac{k}{n}\big)\Big) \quad (10)$$

with $L(\mathbf{y},\mathbf{p},\xi)$ defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$, i.e., $L(\mathbf{y},\mathbf{p},0) = 0$,

$$L(\mathbf{y},\mathbf{p},\xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \xi \leq 1 \quad (11)$$

and $L(\mathbf{y},\mathbf{p},1) = A_{\mathbf{p}}(\mathbf{y})$ thus representing the weighted average. Finally,

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w'_k L\Big(\mathbf{y},\mathbf{p},\frac{k}{n}\Big) \quad (12)$$

with weights

$$\begin{aligned} w'_k &= n(w_{n-k+1} - w_{n-k}) \text{ for } k = 1,\ldots,n-1 \\ w'_n &= nw_1\,. \end{aligned} \quad (13)$$

Graphs of functions $L(\mathbf{y},\mathbf{p},\xi)$ (with respect to ξ) take the form of convex piecewise linear curves, the so-called absolute Lorenz curves [22] connected to the relation of the second order stochastic dominance (SSD). Therefore, formula (12) relates the WOWA average to the SSD consistent risk measures based on the tail means [23] provided that the importance weights are treated as scenario probabilities.

3. The Orness and Risk Preferences

The OWA aggregation may model various preferences from the optimistic (max) to the pessimistic (min). Yager [2] introduced a well appealing concept of the orness measure to characterize the OWA operators. The degree of orness associated with the OWA operator $A_{\mathbf{w}}(\mathbf{y})$ is defined as

$$\text{orness}(\mathbf{w}) = \sum_{i=1}^{m} \frac{m-i}{m-1} w_i\,. \quad (14)$$

For the max aggregation representing the fuzzy OR operator with weights $\mathbf{w} = (1,0,\ldots,0)$ one gets orness$(\mathbf{w}) = 1$ while for the min aggregation representing the fuzzy AND operator with weights $\mathbf{w} = (0,\ldots,0,1)$ one has orness$(\mathbf{w}) = 0$. For the average (arithmetic mean) one gets orness$((1/m,1/m,\ldots,1/m)) = 1/2$. Actually, one may consider a complementary measure of andness defined as andness$(\mathbf{w}) = 1 -$ orness$(\mathbf{w})$. OWA aggregations with orness greater or equal 1/2 are considered or-like whereas the aggregations with orness smaller or equal 1/2 are treated as and-like. The former correspond to rather optimistic preferences while the latter represents rather pessimistic preferences.

The OWA aggregations with monotonic weights are either or-like or and-like. Exactly, decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_m$ define an or-like OWA operator, while increasing weights $w_1 \leq w_2 \leq \ldots \leq w_m$ define an and-like OWA operator. Actually, the orness and the andness properties of the OWA operators with monotonic weights are total in the sense that they remain valid for any subaggregations defined by subsequences of their weights. Namely, for any $2 \leq k \leq m$ one gets

$$\sum_{j=1}^{k} \frac{k-j}{k-1} w_{i_j} \geq \frac{1}{2} \quad \text{and} \quad \sum_{j=1}^{k} \frac{k-j}{k-1} w_{i_j} \leq \frac{1}{2}$$

for the OWA operators with decreasing or increasing weights, respectively. Moreover, the weights monotonicity is necessary to achieve the above total orness and andness properties. Therefore, we will refer to the OWA aggregation with decreasing weights as the totally or-like OWA operator, and to the OWA aggregation with increasing weights as the totally and-like OWA operator.

Yager [24] proposed to define the OWA weighting vectors via the regular increasing monotone (RIM) quantifiers, which provide a dimension independent description of the aggregation. A fuzzy subset Q of the real line is called a RIM quantifier if Q is (weakly) increasing with $Q(0) = 0$ and $Q(1) = 1$. The OWA weights can be defined with a RIM quantifier Q as $w_i = Q(i/m) - Q((i-1)/m)$ and the orness measure can be extended to a RIM quantifier (according to $m \to \infty$) as follows [24]:

$$\text{orness}(Q) = \int_0^1 Q(\alpha)\, d\alpha. \tag{15}$$

Thus, the orness of a RIM quantifier is equal to the area under it. The measure takes the values between 0 (achieved for $Q(1) = 1$ and $Q(\alpha) = 0$ for all other α) and 1 (achieved for $Q(0) = 1$ and $Q(\alpha) = 0$ for all other α). In particular, $\text{orness}(Q) = 1/2$ for $Q(\alpha) = \alpha$ which is generated by equal weights $w_k = 1/n$. Formula (15) allows one to define the orness of the WOWA aggregation (4) which can be viewed with the RIM quantifier $Q(\alpha) = w^*(\alpha)$ [25]. Let us consider piecewise linear function $Q = w^*$ defined by weights vectors $\mathbf{w}$ of dimension n according to the stepwise generation function (8). One may easily notice that decreasing weights $w_1 \geq w_2 \geq \ldots \geq w_n$ generate a strictly increasing concave curve $Q(\alpha) \geq \alpha$ thus guaranteeing the or-likeness of the WOWA operator. Similarly, increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$ generate a strictly increasing convex curve $Q(\alpha) \leq \alpha$ thus guaranteeing the and-likeness of the WOWA operator. Actually, the monotonic weights generate the totally or-like and and-like operators, respectively, in the sense that

$$\int_0^1 \frac{Q(a+\alpha(b-a)) - Q(a)}{Q(b) - Q(a)} d\alpha \geq \frac{1}{2} \tag{16}$$

or

$$\int_0^1 \frac{Q(a+\alpha(b-a)) - Q(a)}{Q(b) - Q(a)} d\alpha \leq \frac{1}{2} \tag{17}$$

for the WOWA operators with decreasing or increasing weights, respectively.

Actually, the absolute Lorenz curve represent a dual characterization of the second stochastic dominance relation [22] which is the most general mathematical model of the risk averse preferences in decisions under risk [26]. Formula (12) represents the WOWA aggregation with increasing preferential weights as the weighted (positive) combination of n tail averages. Therefore, the WOWA objective functions with increasing preferential weights are SSD consistent and they represent the risk averse aggregations of outcomes under several scenarios. Moreover, such WOWA averages may be interpreted as the dual utility criteria within the theory developed by Yaari [27] which was recently reintroduced [28] in a simplified form of the spectral risk measures $\int_0^1 \phi(\xi) F_{\mathbf{y}}^{(-1)}(\xi) d\xi$, where decreasing (nonincreasing) distortion function ϕ represents risk averse preferences. Indeed, according to (6),

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \int_0^1 g(\xi) \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi = \int_0^1 g(1-\xi) F_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$$

thus representing a spectral risk measure with distortion function $\phi(\xi) = g(1-\xi)$, nonincreasing for the increasing weights w_k. Similarly, the generalized WOWA can be expressed with $\phi(\xi) = g_\beta(1-\xi)$ nonincreasing for the relatively increasing weights w_k. As pointed out by Acerbi [28], the subjective risk aversion of a decision maker can be encoded in a function $\phi(\xi)$ defined for all possible $\xi \in (0,1]$ and one cannot see any arbitrary choice of function $\phi(\xi)$. The WOWA aggregations allows one to seek an appropriate function defined by a few preferential weights and possibly breakpoints (for the generalized WOWA).

4. Linear Programming Models

Consider maximization of a risk averse WOWA aggregation defined by increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$

$$\max\{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \ : \ \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\}. \tag{18}$$

Due to formula (12), the problem may be expressed as

$$\max\{\sum_{k=1}^{n} w'_k L(\mathbf{y}, \mathbf{p}, \frac{k}{n}) \ : \ \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\}$$

with positive weights w'_k defined by (13).

According to (11), values of function $L(\mathbf{y}, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{y}, \mathbf{p}, \xi) = \min_{s_i} \{ \sum_{i=1}^{m} y_i s_i : \sum_{i=1}^{m} s_i = \xi; \ 0 \leq s_i \leq p_i, \ \forall\ i \}. \tag{19}$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables.

This difficulty can be overcome by taking advantage of the LP dual to (19). Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{y},\mathbf{p},\xi)$

$$L(\mathbf{y},\mathbf{p},\xi) = \max_{t,d_i} \{ \xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \le y_i,\ d_i \ge 0 \quad \forall\, i \}. \tag{20}$$

Therefore, maximization of the WOWA aggregation (18) can be expressed as follows:

$$\begin{aligned} \max_{t_k,d_{ik},y_i,x_j} \quad & \sum_{k=1}^{n} w'_k \left[\frac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik} \right] \\ \text{s.t.} \quad & t_k - d_{ik} \le y_i,\ d_{ik} \ge 0 \quad \forall\, i,k \\ & \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathscr{F}\,. \end{aligned}$$

Consider multiple criteria problems (1) with linear objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x}$ and polyhedral feasible sets:

$$\max\{(y_1,y_2,\dots,y_m) \ :\ \mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqq \mathbf{0}\}, \tag{21}$$

where $\mathbf{C}$ is an $m \times q$ matrix (consisting of rows $\mathbf{c}_i$), $\mathbf{A}$ is a given $v \times q$ matrix and $\mathbf{b} = (b_1,\dots,b_v)^T$ is a given RHS (right hand side) vector. For such problems, we get the following LP formulation of the WOWA maximization (18):

$$\max_{t_k,d_{ik},y_i,x_j} \quad \sum_{k=1}^{n} \frac{k}{n} w'_k t_k - \sum_{k=1}^{n}\sum_{i=1}^{m} w'_k p_i d_{ik} \tag{22}$$

$$\text{s.t.} \quad \sum_{j=1}^{q} a_{rj} x_j = b_r \qquad r = 1,\dots,v \tag{23}$$

$$y_i - \sum_{j=1}^{q} c_{ij} x_j = 0 \qquad i = 1,\dots,m \tag{24}$$

$$d_{ik} \ge t_k - y_i,\ d_{ik} \ge 0 \qquad i = 1,\dots,m;\ k = 1,\dots,n \tag{25}$$

$$x_j \ge 0 \qquad j = 1,\dots,q \tag{26}$$

Model (22)–(26) is an LP problem with $mn+m+n+q$ variables and $mn+m+v$ constraints. Thus, for problems with not too large number of scenarios (m) and preferential weights (n) it can be solved directly. Note that WOWA model (22)–(26) differs from the analogous deviational model for the OWA optimizations [8] only due to coefficients within the objective function (22) and the possibility of different values of m and n.

The number of constraints in problem (22)–(26) is similar to the number of variables. Nevertheless, for the simplex approach it may be better to deal with the dual of (22)–(26) than with the original problem. Note that variables d_{ik} in the primal are represented with singleton columns. Hence, the corresponding rows in the dual represent only simple upper bounds.

Introducing the dual variables: u_r ($r = 1,\dots,v$), ν_i ($i = 1,\dots,m$) and z_{ik} ($i = 1,\dots,m$; $k = 1,\dots,n$) corresponding to the constraints (23), (24) and (25), respectively, we get the following dual:

$$\begin{aligned} \min_{z_{ik},\nu_i,u_r} \quad & \sum_{r=1}^{v} b_r u_r \\ \text{s.t.} \quad & \sum_{r=1}^{v} a_{rj} u_r - \sum_{i=1}^{m} c_{ij} \nu_i \geqq 0 && j = 1,\dots,q \\ & \nu_i - \sum_{k=1}^{n} z_{ik} \ge 0 && i = 1,\dots,m \\ & \sum_{i=1}^{m} z_{ik} = \frac{k}{n} w'_k && k = 1,\dots,n \\ & 0 \le z_{ik} \le p_i w'_k && i = 1,\dots,m;\ k = 1,\dots,n \end{aligned} \tag{27}$$

The dual problem (27) contains: $m+n+q$ structural constraints, $m+v$ unbounded variables and mn bounded variables. Since the average complexity of the simplex method depends on the number of constraints, the dual model (27) can be directly solved for quite large values of m and n. Moreover, the columns corresponding to mn variables z_{ik} form the transportation/assignment matrix thus allowing one to employ special techniques of the simplex SON (special ordered network) algorithm [29] for implicit handling of these variables. Such techniques increase dramatically efficiency of the simplex method but they require a special tailored implementation. We have not tested this approach within our initial computational experiments based on the use of a general purpose LP code.

5. Computational Tests

In order to analyze the computational performances of the LP model for the WOWA optimization, similarly to [8], we have solved randomly generated problems of portfolio optimization according to the (discrete) scenario analysis approach [6]. There is given a set of securities for an investment $J = \{1,2,\dots,q\}$. We assume, as usual, that for each security $j \in J$ there is given a vector of data $(c_{ij})_{i=1,\dots,m}$, where c_{ij} is the observed (or forecasted) rate of return of security j under scenario i (hereafter referred to as outcome). We consider discrete distributions of returns defined by the finite set $I = \{1,2,\dots,m\}$ of scenarios with the assumption that each scenario can be assigned the importance weight p_i that can be seen as the subjective probability of the scenario. The outcome data forms an $m \times q$ matrix $\mathbf{C} = (c_{ij})_{i=1,\dots,m;j=1,\dots,q}$ whose columns correspond to securities while rows $\mathbf{c}_i = (c_{ij})_{j=1,2,\dots,q}$ correspond to outcomes. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,q}$ denote the vector of decision variables defining a portfolio. Each variable x_j expresses the portion of the capital invested in the corresponding security. Portfolio $\mathbf{x}$ generates outcomes

$$\mathbf{y} = \mathbf{C}\mathbf{x} = (\mathbf{c}_1\mathbf{x}, \mathbf{c}_2\mathbf{x}, \dots, \mathbf{c}_m\mathbf{x})\,.$$

The portfolio selection problem can be considered as an LP problem with m uniform objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{q} c_{ij}x_j$ to be maximized [6]:

$$\max \{\mathbf{Cx} : \sum_{j=1}^{q} x_j = 1;\ x_j \geq 0,\ j = 1,\dots,q\}.$$

Hence, our portfolio optimization problem can be considered a special case of the multiple criteria problem and one may seek an optimal portfolio with some criteria aggregation. Note that the aggregation must take into account the importance of various scenarios thus allowing importance weights p_i to be assigned to several scenarios. Further the preferential weights w_k must be increasing to represent the risk averse preferences (more attention paid on improvement of smaller outcomes). Thus we get the WOWA maximization problem:

$$\max \{A_{\mathbf{w},\mathbf{p}}(\mathbf{Cx}) : \sum_{j=1}^{q} x_j = 1;\ x_j \geq 0,\ j = 1,\dots,q\}. \quad (28)$$

Our computational tests were based on the randomly generated problems (28) with varying number q of securities (decision variables) and number m of scenarios. The generation procedure worked as follows. First, for each security j the maximum rate of return r_j was generated as a random number uniformly distributed in the interval $[0.05, 0.15]$. Next, this value was used to generate specific outcomes c_{ij} (the rate of return under scenarios i) as random variables uniformly distributed in the interval $[-0.75r_j, r_j]$. Further, strictly increasing and positive weights w_k were generated. The weights were not normalized which allowed us to define them by the corresponding increments $\delta_k = w_k - w_{k-1}$. The latter were generated as uniformly distributed random values in the range of 1.0 to 2.0, except from a few (5 on average) possibly larger increments ranged from 1.0 to $n/3$. Importance weights p_i were generated according to the exponential smoothing scheme, which assigns exponentially decreasing weights to older or subjectively less probable scenarios: $p_i = \alpha(1-\alpha)^{i-1}$ for $i = 1,2,\dots,m$ and the parameter α is chosen for each test problem size separately to keep the value of p_m around 0.001.

We tested solution times for different size parameters m and q. The basic tests were performed for the standard WOWA model with $n = m$. However, we also analyzed the case of larger n for more detailed preferences modeling, as well as the case of smaller n thus representing a rough preferences model. For each number of securities q and number of criteria (scenarios) m we solved 10 randomly generated problems (28). All computations were performed on a PC with the Athlon 64, 1.8 GHz processor employing the CPLEX 9.1 package. The 600 seconds time limit was used in all the computations.

In Tables 1 and 2 we show the solution times for the primal (22)–(26) and the dual (27) forms of the computational model, being the averages of 10 randomly generated problems. Upper index in front of the time value indicates the number of tests among 10 that exceeded the time limit. The empty cell (minus sign) shows that this occurred for all 10 instances. Both forms were solved by the CPLEX code without taking advantages of the constraints structure specificity. The dual form of the model performs much better in each tested problem size. It behaves very well with increasing number of securities if the number of scenarios does not exceed 100. Similarly, the model performs very well with increasing number of scenarios if only the number of securities does not exceed 50.

Table 1

Solution times [s] for the primal model (22)–(26)

Scenarios	Number of securities (q)							
(m)	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.8	1.0	1.4	1.7	1.7	1.7	1.6	1.7
100	21.0	29.0	34.1	41.8	51.9	70.0	95.4	86.9
150	187.0	243.6	312.9	354.2	402.7	474.8	^{1}474.9	^{6}562.2

Table 2

Solution times [s] for the dual model (27)

Scenarios	Number of securities (q)							
(m)	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.1
50	0.0	0.0	0.3	0.3	0.4	0.4	0.5	0.6
100	0.4	0.6	1.5	6.7	8.4	10.2	11.6	13.4
150	1.3	2.0	3.8	24.2	49.0	59.2	62.1	62.7
200	3.0	4.1	8.7	66.6	144.8	225.0	243.0	246.9
300	9.7	14.3	31.0	^{1}291.4	^{4}491.3	–	–	–
400	22.8	34.4	82.2	^{4}344.1	^{7}555.1	–	–	–

Table 3

Solution times [s] for different numbers of preferential weights $(q = 50)$

Number of	Number of preferential weights (n)									
scenarios (m)	3	5	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.1
20	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.3	0.3
50	0.0	0.0	0.0	0.0	0.3	0.2	0.4	0.6	1.1	1.6
100	0.0	0.0	0.1	0.3	1.0	1.5	1.5	2.1	3.7	5.1
150	0.0	0.1	0.2	0.5	3.7	4.5	3.8	4.6	7.7	11.0
200	0.1	0.1	0.3	1.2	8.0	11.6	7.1	8.7	13.8	23.7
300	0.1	0.3	0.7	3.5	19.6	29.1	12.2	16.9	31.0	41.2
400	0.2	0.4	1.6	6.5	36.6	48.6	16.4	28.4	45.8	82.2

Table 3 presents solution times for different numbers of the preferential weights. The number of securities equals 50. It can be noticed that increasing the number of preferential weights and thus the number of breakpoints in the interpolation function induce moderate increase in the computational complexity. On the other hand, the computational efficiency can be significantly improved by reducing the number of preferential weights to a few which can be reasonable in non-automated decision making support systems.

6. Concluding Remarks

The WOWA aggregation [12] represents a universal tool allowing one to combinine outcomes under several scenarios to form overall objective functions taking into account both the risk aversion preferences depicted with the preferential weights allocated to ordered outcomes as well as the scenarios importance expressed with weights allocated to several scenarios. The ordering operator used to define the WOWA aggregation is, in general, hard to implement within optimization problems. We have shown that the risk averse WOWA aggregations are characterized by the increasing weights and their optimization can be modeled by introducing auxiliary linear constraints. Hence, an LP decision under risk problem with the risk averse WOWA aggregation of outcomes under several scenarios can be formed as a standard linear program. Moreover, it can be further simplified by taking advantages of the LP duality.

Our computational experiments show that the LP formulation enables to solve effectively medium size WOWA problems. Actually, the number of few hundred scenarios efficiently covered by the dual LP model in less a minute for problems with limited number of structural variables seems to be quite enough for most applications to decisions under risk. The problems have been solved directly by a general purpose LP code. Taking advantages of the constraints structure specificity may remarkably extend the solution capabilities. In particular, the simplex SON algorithm [29] may be used for exploiting the LP embedded network structure in the dual form of the model. This seems to be a very promising direction for further research.

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