Paper

Coherence of radial implicative fuzzy systems with nominal consequents

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Abstract— In the paper we are interested in the question of coherence of radial implicative fuzzy systems with nominal consequents (radial I-FSs with NCs). Implicative fuzzy systems are fuzzy systems employing residuated fuzzy implications for representation of IF-THEN structure of their rules. Radial fuzzy systems are fuzzy systems exhibiting the radial property in antecedents of their rules. The property simplifies computational model of radial systems and makes the investigation of their properties more tractable. A fuzzy system has nominal consequents if its output is defined on a finite unordered set of possible actions which are generally quantitatively incomparable. The question of coherence is the question of under which conditions we are assured that regardless the input to the system is, an output of the system exists, i.e., the output is non-empty. In other words, a fuzzy system is coherent if it has no contradictory rules in its rule base. In the paper we state sufficient conditions for a radial I-FS with NCs to be coherent.

Keywords— implicative fuzzy system, radial fuzzy system, nominal output space, coherence.

1. Introduction

In the theory of fuzzy systems there are generally recognized two approaches to the representation of IF-THEN rules and their groups – rule bases [3, 7, 8]. They are the *conjunctive* and the *implicative* approach. In the conjunctive approach, IF-THEN structure of a rule is represented by a fuzzy conjunction and individual rules are combined by a fuzzy disjunction. In the implicative approach, IF-THEN structure of a rule is represented by a fuzzy implication and individual rules are combined by a fuzzy conjunction.

Radial fuzzy systems are fuzzy systems which have membership functions of fuzzy sets in their rules represented by radial functions and exhibit the radial property. The radial property is the shape preservation property related to antecedents (IF parts) of IF-THEN rules. The presence of this property simplifies the computational model of radial fuzzy systems and enables an effective study of their properties.

Fuzzy systems with nominal consequents (THEN parts) are those systems with finite, generally unordered, output spaces. Such a space forms the universe of discourse on which fuzzy sets in consequents are specified. Such an universe can be treated as a set of possible actions which are quantitatively incomparable.

The question of coherence of a fuzzy system is an important question related mainly to the theory of implicative fuzzy systems [4, 9]. The request for coherence of an implicative system can be seen as the request for the non-presence of contradictory rules in its rule base, for if the rules are contradictory there exists an input making the output of the system to be empty. As a typical example of contradictory rules consider the situation if (simultaneously) one rule indicates *go left* action and the other *go right* action.

In the literature, the question of coherence was discussed mainly for fuzzy systems with ordinal consequents [2, 4], i.e., for the systems having consequents' fuzzy sets specified on ordered universes of discourse, typically on real line $\mathcal{R}$.

In this paper we are interested in the study of coherence of radial fuzzy systems with nominal consequents (radial I-FSs with NCs). In the next section we introduce the computational model of these systems in an explicit way. Section 3 is the main section of the paper and contains two theorems stating sufficient conditions for coherence of a radial I-FS with NCs. The paper concludes by Section 4.

2. Radial I-FSs with NCs

We consider the standard architecture of a fuzzy system. That is, the system consists of four building blocks – *singleton* fuzzifier, implicative rule base, compositional rule of inference (CRI) inference engine, and a defuzzification block. The flow of a signal is as usual, i.e., from the fuzzifer to the defuzzifier through the inference engine [7, 8].

2.1. Computational model of I-FSs with NCs

Under the implicative approach, a rule base consisting of m, $m \in \mathcal{N} = \{1, 2, \dots\}$, rules has the following mathematical representation:

$$RB(\boldsymbol{x}, y) = \bigwedge_{j=1}^{m} A_j(\boldsymbol{x}) \to B_j(y), \tag{1}$$

where A_j is the representation of the antecedent in the jth rule, $j = 1, \dots, m$, B_j is the consequent fuzzy set, $\to$ is a *residuated* fuzzy implication and $\bigwedge$ a fuzzy conjunction. Typically $\bigwedge = \star$, where $\star$ is the t-norm which is used to form antecedents of rules and for specification of $\to$, see below.

Antecedents A_js are generally specified on n-dimensional, $n \in \mathcal{N}$, input space $X = \mathcal{R}^n$ and represented in the standard way as

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n), \tag{2}$$

where $\boldsymbol{x} \in \mathcal{R}^n$, $\boldsymbol{x} = (x_1, \dots, x_n)$, A_{ji}, $i = 1, \dots, n$, are one-dimensional fuzzy sets defined on respective one-dimensional parts of input space (in fact these are real lines $\mathcal{R}$), and $\star$ is a t-norm representing a fuzzy conjunction (and linguistic connective). As mentioned, this $\star$ is usually also used for representing $\bigwedge$ in Eq. (1).

As we are interested in fuzzy systems with nominal consequents we consider output space Y to correspond to an *unordered finite* set (nominal space) of $l \in \mathcal{N}$ generally mutually incomparable actions y_k, $k = 1, \ldots, l$, i.e., $Y = \{y_1, \ldots, y_l\}$. Consequents' fuzzy sets B_js are then considered to be specified on Y.

Let the input to a fuzzy system be $\boldsymbol{x}^* \in \mathcal{R}^n$. As we consider the singleton fuzzifier to be employed in the system the general CRI formula for computing output fuzzy set B' from the fuzzy system is simplified into the form

$$B'(y) = RB(\boldsymbol{x}^*, y)\,, \tag{3}$$

where RB is the representation of rule base. Employing the implicative rule base Eq. (1) the above reads as

$$B'(y_k) = \bigwedge_{j=1}^{m} A_j(\boldsymbol{x}^*) \rightarrow B_j(y_k). \tag{4}$$

Introducing m individual output fuzzy set B'_j, each related to the single rule j and defined by

$$B'_j(y_k) = A_j(\boldsymbol{x}^*) \rightarrow B'_j(y_k), \tag{5}$$

the overall output is specified as

$$B'(y_k) = B'_1(y_k) \star \cdots \star B'_m(y_k). \tag{6}$$

To proceed let us recall the concept of residuated implication and its properties. Residuated fuzzy implications are generalizations of Boolean implications. A residuated implication $\rightarrow_\star$ is derived on the basis of its associated t-norm $\star$ according to formula

$$a \rightarrow_\star b = \sup\{c \in [0,1] \,|\, a \star c \leq b\}. \tag{7}$$

Examples of residuated fuzzy implications are the *Gödel implication* derived from the minimum t-norm: $a \rightarrow_{\mathrm{M}} b = 1$ iff $a \leq b$ and $a \rightarrow_{\mathrm{M}} b = b$ iff $a > b$; and the *Goguen implication* derived from the product t-norm: $a \rightarrow_{\mathrm{P}} b = 1$ iff $a \leq b$ and $a \rightarrow_{\mathrm{P}} b = b/a$ iff $a > b$. For details about residuated implications see [6, 7].

An important property, valid for any residuated implication $\rightarrow$ (in the sequel we will not explictly indicate the associated t-norm $\star$), is

$$a \rightarrow b = 1 \quad \text{iff} \quad a \leq b. \tag{8}$$

On the basis of this property the computational model of an I-FS with NCs is stated as follows:

Let the consequents fuzzy sets B_js be normal, i.e., for each rule j there exists a k such that $B_j(y_k) = 1$. Then the core (or kernel) of B'_j set is specified as $core(B'_j) = \{y_k \,|\, B'_j(y_k) = 1\}$. Due to the normality of B_j, $core(B'_j) \neq \emptyset$. Going back to how B'_j sets are defined, formula (5), and employing the property (8) we can see that depending on value of $A_j(\boldsymbol{x}^*)$ another y_k(s) can occur in $core(B'_j)$. More specifically, for a given input $\boldsymbol{x}^*$, an $y_k \in Y$ is in $core(B'_j)$ if and only if $A_j(\boldsymbol{x}^*) \leq B_j(y_k)$. Let us denote $core(B'_j)$ for a given input $\boldsymbol{x}^*$ by $I_j(\boldsymbol{x}^*)$. The following specification formula for $I_j(\boldsymbol{x}^*)$ can be adopted:

$$I_j(\boldsymbol{x}^*) = \{\, y_k \in Y \,|\, A_j(\boldsymbol{x}^*) \leq B_j(y_k) \,\}. \tag{9}$$

With respect to the overall output B' of an I-FS with NCs, which is given by formula (6), let us assume that for a given input $\boldsymbol{x}^*$ the corresponding output fuzzy set B' is normal and let us denote its core by $I(\boldsymbol{x}^*)$. From the properties of t-norms ($x_1 \star \ldots \star x_n = 1$ iff $x_i = 1$ for all i) we obtain $I_j(\boldsymbol{x}^*)$ to be determined as the intersection of particular cores $I_j(\boldsymbol{x}^*)$, i.e.,

$$I(\boldsymbol{x}^*) = \bigcap_{j=1}^{m} I_j(\boldsymbol{x}^*). \tag{10}$$

If $I(\boldsymbol{x}^*) \neq \emptyset$, then $y_k \in I(\boldsymbol{x}^*)$ are those actions from Y which are fully consistent with the input $\boldsymbol{x}^*$ under the given implicative rule base. That is, they make the evaluation of all rules in the rule base to be (simultaneously) 1, so they are natural candidates for the output of the I-FS for the given input $\boldsymbol{x}^* \in \mathcal{R}^n$.

Let as assume fuzzy set B' to be normal for any input $\boldsymbol{x}^*$ and therefore $core(B') = I(\boldsymbol{x}^*) \neq \emptyset$ for any $\boldsymbol{x}^*$. If we take as the output of I-FS with NCs an element from $I(\boldsymbol{x}^*)$ we can consider this process as defuzzification of B' set. The answer to the question of which concrete action from $I(\boldsymbol{x}^*)$ is taken (what deffuzification method is used) depends on concrete application. Here we will consider as output for given input $\boldsymbol{x}^* \in \mathcal{R}^n$ the whole set $I(\boldsymbol{x}^*)$. Formally written, the computational model of an I-FS with NCs has the form

$$\mathrm{I-FSNC}(\boldsymbol{x}^*) = I(\boldsymbol{x}^*) = \bigcap_{j=1}^{m} I_j(\boldsymbol{x}^*), \tag{11}$$

$$\mathrm{I-FSNC}(\boldsymbol{x}^*) = \bigcap_{j=1}^{m} \{\, y_k \,|\, A_j(\boldsymbol{x}^*) \leq B_j(y_k) \,\}. \tag{12}$$

Let us show that for the computation of an I-FS (with NCs) only firing rules are important. Indeed, let $\boldsymbol{x}^*$ be an input into the system, then two cases are possible with respect to the jth rule: either the rule does not fire, i.e., $A_j(\boldsymbol{x}^*) = 0$ or it fires, i.e., $A_j(\boldsymbol{x}^*) > 0$.

With respect to the first case of $A_j(\boldsymbol{x}^*) = 0$, we get immediately $I_j(\boldsymbol{x}^*) = Y$ on the basis of property (8) of residuated implication ($a = 0$). Forming the final output $I(\boldsymbol{x}^*)$ by intersection (11), we see that if we exclude $I_j(\boldsymbol{x}^*)$ from the intersection, then the result remains the same. Thus, if a rule in an I-FS does not fire then it can be excluded from the computation, under the assumption that at least one another rule fires. If none of rules fires, i.e., if $A_j(\boldsymbol{x}^*) = 0$ for all j, then $I(\boldsymbol{x}^*) = Y$.

With respect to general formula (11) there are two questions important. The first is related to how to specify $I_j(\boldsymbol{x}^*)$ sets in an explicit way. The second is related to the assumption of non-emptiness of $I(\boldsymbol{x}^*)$ for any $\boldsymbol{x}^* \in \mathcal{R}^n$. The first question can be answered more explicitly in connection with the class of so called radial fuzzy systems. The second relates to the concept of coherence and for radial systems is treated in Section 3. Let us now introduce the class of radial fuzzy systems.

2.2. Radial I-FSs with NCs

The concept of a radial implicative fuzzy system with nominal consequents is defined as follows:

Definition 1: An implicative fuzzy system with nominal consequents is *radial* if:

(1) There exists a continuous function $act : [0,+\infty) \to [0,1]$, $act(0) = 1$ as follows: (a) either there exists $z_0 \in (0,+\infty)$ such that act is strictly decreasing on $[0,z_0]$ and $act(z) = 0$ for $z \in [z_0,+\infty)$ or (b) act is strictly decreasing on $[0,+\infty)$ and $\lim_{z\to+\infty} act(z) = 0$.

(2) Fuzzy sets in antecedent and consequent parts of the jth rule are specified as

$$A_{ji}(x_i) = act\left(\left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right), \qquad (13)$$

$$B_j(y_k) = \mu_{kj}, \qquad (14)$$

where for each B_j there exists at least one action y_k such that $\mu_{kj} = 1$, i.e., fuzzy sets B_js are normal; $n,m,l \in \mathcal{N}$; $i,j,k = 1,\ldots,n,m,l$, respectively; $\boldsymbol{x} \in \mathcal{R}^n$, $\boldsymbol{x} = (x_1,\ldots,x_n)$; $y_k \in Y = \{y_1,\ldots,y_l\}$; $\boldsymbol{a}_j \in \mathcal{R}^n$, $\boldsymbol{a}_j = (a_{j1},\ldots,a_{jn})$; $\boldsymbol{b}_j \in \mathcal{R}^n_+$, $\boldsymbol{b}_j = (b_{j1},\ldots,b_{jn})$, (i.e., $b_{ji} > 0$); $\mu_{kj} \in [0,1]$.

(3) For each $\boldsymbol{x} \in \mathcal{R}^n$ the *radial property* holds, i.e.,

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) = act(||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}), \qquad (15)$$

where $||\cdot||_{\boldsymbol{b}_j}$ is a scaled version of some ℓ_p norm in $\mathcal{R}^n$. This norm is common to all rules of the fuzzy system.

Let us comment on the definition to clarify the concept. An I-FS with NCs is radial if it satisfies three requirements. Before we dicuss these requirements let us recall the concepts of radial function and ℓ_p norm, both defined in $\mathcal{R}^n$ space.
Radial functions are generally defined by formula $f(\boldsymbol{x}) = \Phi(||\boldsymbol{x} - \boldsymbol{a}||)$, where Φ is a function from $[0,+\infty)$ to $\mathcal{R}$, $||\cdot||$ is a norm in $\mathcal{R}^n$ and $\boldsymbol{a} \in \mathcal{R}^n$ is a central point of the function. Concerning radial fuzzy systems, the class of so-called ℓ_p norms in $\mathcal{R}^n$ is important [5]. The definition formula of ℓ_p norms depends on parameter $p \in [1,+\infty]$ and reads as follows:

$$\begin{aligned} ||\boldsymbol{u}||_p &= (|u_1|^p + \cdots + |u_n|^p)^{1/p} \quad \text{for} \quad p \in [1,+\infty), \\ ||\boldsymbol{u}||_\infty &= \lim_{p\to+\infty} ||\boldsymbol{u}||_p = \max_i\{|u_i|\}. \end{aligned} \qquad (16)$$

Scaled ℓ_p norms, denoted by $||\cdot||_{p_{\boldsymbol{b}}}$, are derived from corresponding ℓ_p norms by incorporating a vector $\boldsymbol{b} \in \mathcal{R}^n_+$ of scaling parameters, $\boldsymbol{b} = (b_1,\ldots,b_n)$, $b_i > 0$, into the above formulas. That is,

$$\begin{aligned} ||\boldsymbol{u}||_{p_{\boldsymbol{b}}} &= (|u_1/b_1|^p + \ldots + |u_n/b_n|^p)^{1/p}\,;\ p \in [1,+\infty), \\ ||\boldsymbol{u}||_{\infty_{\boldsymbol{b}}} &= \lim_{p\to+\infty} ||\boldsymbol{u}||_{p_{\boldsymbol{b}}} = \max_i\left\{|u_i/b_i|\right\}. \end{aligned} \qquad (17)$$

Clearly, original unscaled ℓ_p norms are obtained from scaled ones by choosing $\boldsymbol{b} = \mathbf{1} = (1,\ldots,1)$. The most prominent examples of scaled ℓ_p norms are scaled octaedric ($p = 1$), Euclidean ($p = 2$) and cubic ($p = +\infty$) norms.

Now we can discuss the definition of a radial fuzzy system. The first two requirements are related to the specification of membership functions of fuzzy sets employed in IF-THEN rules. Especially, they relate to the shapes of one-dimensional fuzzy sets which form antecedents of rules.

The requirement (1) specifies the "shape" of one-dimensional fuzzy sets by specification of an *act* function. This function is considered to be generally non-increasing and can have two variants. The first variant corresponds to a strictly decreasing function, the other has strictly decreasing part and after reaching zero it is constant.
The requirement (2) is in fact the prescription which makes one-dimensional fuzzy sets A_{ji} to be radial. In one-dimensional space the norm correspond to the absolute value, central point corresponds to $a_{ji} \in \mathcal{R}$ and also (width) scaling parameter $b_{ji} \in \mathcal{R}_+$, ($\mathcal{R}_+ = (0,\infty)$, i.e., $b_{ji} > 0$) is employed. The shape is determined by *act* function.
Consequents' fuzzy sets B_js are required to be normal. In formula (14) the simplification of notation is adopted in form of $B_j(y_k) = \mu_{kj}$ (the indices are switched). Particular μ_{kj} can be seen as the membership degree of action y_k to the consequent of the jth rule.
The requirement (3) is in fact the radial property. The property requires a radial shape preservation of one-dimensional fuzzy sets in antecedents after their combination by a t-norm according to formula (2). Mathematically, the property is specified by equality (15). We can see that the property requires antecedents to be represented by radial functions (now in n-dimensional space $\mathcal{R}^n$) which have the same shape *act* as one-dimensional fuzzy sets A_{ji}. Moreover, central point $\boldsymbol{a}_j \in \mathcal{R}^n$ is required to be composed from one-dimensional central points a_{ji}, i.e., $\boldsymbol{a}_j = (a_{j1},\ldots,a_{jn})$. Similarly, scaling parameter $\boldsymbol{b}_j \in \mathcal{R}^n_+$ is required to be composed from one-dimensional scaling parameters b_{ji}, i.e., $\boldsymbol{b}_j = (b_{j1},\ldots,b_{jn})$.
The radial property is not trivial. If the specification of one-dimensional fuzzy sets A_{ji} is given together with a certain t-norm, then the specification of A_j is determined; and this specification need not have the form of a multidimensional radial function. For example, if triangular fuzzy sets are combined by the product t-norm then it can be shown [2] that the resulting representation of A_j does not exhibit the radial property in the sense of formula (15).
The question of which shapes (*act* functions) and t-norms can be combined so the radial property hold is partially

answered in [1, 2]. As an example of radial I-FSs let us present here Mamdani and Gaussian radial I-FSs [2].

In the Mamdani radial I-FS, the used t-norm is the minimum t-norm, corresponding residuated implication is the Goguen implication and *act* function has form $act(z) = \max\{0, 1-z\}$. This *act* function is of (1)(a) type of Definition 1. The resulting one-dimensional fuzzy sets are triangular and ℓ_p norm in antecedents is the cubic norm.

In the Gaussian radial I-FS, the used t-norm is the product t-norm, corresponding residuated implication is the Gödel implication and *act* function has form $act(z) = \exp(-z^2)$. This *act* function is of (1)(b) type of Definition 1. The resulting one-dimensional fuzzy sets are Gaussian curves and ℓ_p norm in antecedents is the Euclidean norm. In the case of this system the radial property can be easily verified on the basis of well know behavior of Gaussian curves with respect to product (the product of one-dimensional Gaussian curves is a multidimensional Gaussian curve).

2.3. Computational model of radial I-FSs with NCs

In the previous section we have presented the notion of a radial I-FS with nominal consequents. In this section we will discuss its computational model in a more explicit way.

As we have mentioned in Subsection 2.1, with respect to the general computational model of an I-FS with NCs there are important two questions. The first relates to the specification of particular outputs $I_j(\boldsymbol{x}^*)$ and the second to the coherence of the system. In the case of a radial I-FS with NCs we have the following straightforward answer to the first question: if the system is radial, then

$$I_j(\boldsymbol{x}^*) = \{\, y_k \mid act(\|\boldsymbol{x}^* - \boldsymbol{a}_j\|)_{\boldsymbol{b}_j} \le \mu_{kj} \,\}. \tag{18}$$

Thus, on the basis of the radial property, a $y_k \in I_j(\boldsymbol{x}^*)$ if and only if a transformed (by *act* function) scaled norm of distance of input $\boldsymbol{x}^*$ from central point $\boldsymbol{a}_j$ is lower or equal to the membership degree of y_k to the consequent fuzzy set B_j.

In fact, the computational gain from the presence of radial property is not so significant as for the radial I-FSs with ordinal consequents [2]. However, the radial property allows us to explicitly express for which inputs $\boldsymbol{x}^* \in \mathcal{R}^n$ the action y_k is not included in $I_j(\boldsymbol{x}^*)$. Based on formula (18), we know that $y_k \notin I_j(\boldsymbol{x}^*)$ iff $act(\|\boldsymbol{x}^* - \boldsymbol{a}_j\|)_{\boldsymbol{b}_j} > \mu_{kj}$. Now, the following chain of equivalent inequalities can be introduced: let for an input $\boldsymbol{x}^*$ the action $y_k \notin I_j(\boldsymbol{x}^*)$, then

$$act(\|\boldsymbol{x}^* - \boldsymbol{a}_j\|_{\boldsymbol{b}_j}) \;>\; \mu_{kj}, \tag{19}$$

$$act_+(\|\boldsymbol{x}^* - \boldsymbol{a}_j\|_{\boldsymbol{b}_j}) \;>\; \mu_{kj}, \tag{20}$$

$$\|\boldsymbol{x}^* - \boldsymbol{a}_j\|_{\boldsymbol{b}_j} \;<\; act_+^{-1}(\mu_{kj}), \tag{21}$$

$$\|\boldsymbol{x}^* - \boldsymbol{a}_j\|_{\boldsymbol{b}_j} \;<\; r_{kj}, \tag{22}$$

where $r_{kj} = act_+^{-1}(\mu_{kj})$. The reverse holds too, i.e., if (22) holds then $y_k \notin I_j(\boldsymbol{x}^*)$.

In the above chain of inequalities, if *act* is of type (1)(a) of Definition 1, then act_+ is the restriction of *act* function on interval $[0, z_0]$. This restriction is a strictly decreasing function and therefore $act_+^{-1} : [0,1] \to [0, z_0]$ is well defined. If *act* function is of type (1)(b), then $act_+ = act$ for $z \in [0, +\infty)$ and $act_+^{-1} = act^{-1}$ on domain $(0,1]$. We set by definition $act_+^{-1}(0) = +\infty$. Thus, also in this case $act_+^{-1} : [0,1] \to [0, +\infty]$ is well defined. Based on the definition of act_+^{-1} function we see that values r_{kj} are well defined too, and $r_{kj} \in [0, +\infty]$.

As we will see in the next section, the possibility of introduction of inequality (22), which would not be possible without presence of the radial property, helps significantly in testing the coherence of radial I-FSs with NCs.

3. Coherence of radial I-FSs with NCs

The question of coherence of an implicative fuzzy system is the question of non-presence of contradictory rules in the rule base of the system. Incoherence is indicated by empty output of the system for certain input(s). The emptiness is caused by non-existence of common points in outputs of individual rules in the rule base (the intersection of particular outputs is empty). In order to avoid this situation we are looking for at least sufficient conditions on parameters of the system which assure that the case of empty intersection cannot occur for any possible input. Thus, we can say that the system is coherent if and only if for any possible input it has a non-empty output. In this section we will investigate the coherence of radial I-FSs with NCs. In order to obtain sufficient conditions we start from the computational model of this class of systems.

Based on chain of inequalities (19)–(22) we can state for every action y_k and rule j so-called *region of incoherence* RIC_{kj} as the set of those inputs[1] $\boldsymbol{x} \in \mathcal{R}^n$ for which $y_k \notin I_j(\boldsymbol{x})$. We have

$$RIC_{kj} = \{\boldsymbol{x} \in \mathcal{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}_j\|_{\boldsymbol{b}_j} < r_{kj}\,\}. \tag{23}$$

In other words, RIC_{kj} is the set of those inputs which exclude action y_k from output $I_j(\boldsymbol{x})$, or, it is the set of those inputs for which action y_k is not included in output $I_j(\boldsymbol{x})$. Clearly, if $\boldsymbol{x} \notin RIC_{kj}$, then $y_k \in I_j(\boldsymbol{x})$. Based on the above formula, the regions of incoherence RIC_{kj}s, $k = 1, \ldots, l$, $j = 1, \ldots, m$ can be seen as deformed (due to the scaling in norm) hyperballs in $\mathcal{R}^n$ space.

For every action y_k, $k = 1, \ldots, l$, let us introduce its region of incoherence RIC_k as the union of RIC_{kj} over all rules j, i.e.,

$$RIC_k = \bigcup_{j=1}^{m} RIC_{kj}. \tag{24}$$

RIC_k can be interpreted as follows: if an input $\boldsymbol{x}$ is in RIC_k, then there exists a rule j such that $y_k \notin I_j(\boldsymbol{x})$ and therefore y_k is not included in the overall output $I(\boldsymbol{x})$ of the system. The reverse holds too, i.e., if $y_k \notin I(\boldsymbol{x})$ then $\boldsymbol{x} \in RIC_k$.

[1] From now on we will denote the input by $\boldsymbol{x}$ instead of former $\boldsymbol{x}^*$.

The next step in specification of conditions of coherence is straightforward.

Lemma 1: A radial I-FS with NCs is coherent if and only if

$$RIC = \bigcap_{k=1}^{l} RIC_k = \bigcap_{k=1}^{l} \bigcup_{j=1}^{m} RIC_{kj} = \emptyset. \qquad (25)$$

Proof: Inspecting the intersection of RIC_k over possible actions two cases can occur. If this intersection is empty, then there does not exist any input which would excluded simultaneously all actions from the output of the system, i.e., the system is coherent. On the other hand, if the intersection is non-empty then points (inputs) in this intersection are witnesses of incoherence, as they exclude simultaneously all actions from the overall output. □

In the sequel we will investigate the intersection of unions of deformed hyperballs presented by formula (25). Actually, the question is how to test that the intersection of unions of hyperballs is empty (or non-empty). To solve this question, let us explicitly remark that if $\mu_{kj} = 1$, then $r_{kj} = 0$ (because $act_+^{-1}(1) = 0$) and formula (23) yields $RIC_{kj} = \emptyset$. On the other hand, if $\mu_{kj} = 0$ and *act* is of (1)(b) type of Definition 1, then $r_{kj} = +\infty$ and $RIC_{kj} = \mathcal{R}^n$. Both cases are important as we will see below.

Intersection (25) is non-empty (the system is incoherent) if and only if there exists a permutation with repetition $\pi = (j_1,\ldots,j_l)$, $\pi(1) = j_1, \pi(2) = j_2,\ldots$ of rule indices $\{1,\ldots,m\}$ such that the intersection

$$I_\pi = RIC_{1,j_1} \cap RIC_{2,j_2} \cap \cdots \cap RIC_{l,j_l} \qquad (26)$$

is non-empty. The length of the permutation is l, i.e., it equals to the number of actions. If we show that for any permutation π with repetition of length l from rule indices $\{1,\ldots,m\}$ the intersection (26) is empty, then the corresponding radial I-FS with NCs is coherent.

To better understand the above, it is worth to consider the scheme presented in Table 1. If the system is incoherent, then there exists an input $\boldsymbol{x}$ such that simultaneously $\boldsymbol{x} \in RIC_k$ for all k. Since RIC_k is given by union of RIC_{kj}, this can be interpreted as follows: for this $\boldsymbol{x}$ and for every row k in Table 1 there exists a column j such that $\boldsymbol{x} \in RIC_{kj}$. We code the indices of these columns as permutation $\pi(k)$. Clearly, if for every such permutation π the intersection $\bigcap_k RIC_k$ is empty then the system is coherent. This kind of testing of coherence will be elaborated in the sequel.

Table 1
Incoherence regions

	$j=1$	$j=2$	$\ldots$	$j=m$	$\bigcup_j RIC_{kj}$
$k=1$	RIC_{11}	RIC_{12}	$\ldots$	RIC_{1m}	RIC_1
$k=2$	RIC_{21}	RIC_{22}	$\ldots$	RIC_{2m}	RIC_2
			$\vdots$		
$k=l$	RIC_{l1}	RIC_{l2}	$\ldots$	RIC_{lm}	RIC_l
					$\bigcap_k RIC_k$

There are two problems related to the testing scheme proposed. First, how to test the emptiness of intersection (26) for a given permutation π. Second, how to cope with the curse of dimensionality as the number of all permutations is m^l for given number of rules m and actions l.

Let I_π of (26) be non-empty for given permutation π, $\pi(k) = j_k$, $k = 1,\ldots,l$, i.e., $\pi = (j_1,\ldots,j_l)$. Then for $\boldsymbol{x} \in I_\pi$ we have $\boldsymbol{x} \in RIC_{k,\pi(k)}$ for all k, which yields the following k inequalities:

$$||\boldsymbol{x} - \boldsymbol{a}_{\pi(k)}||_{\boldsymbol{b}_{\pi(k)}} < r_{k,\pi(k)} \quad \text{for} \quad k = 1,\ldots,l. \qquad (27)$$

To proceed let us remark that for any scaled ℓ_p norm in $\mathcal{R}^n$ and any vector $\boldsymbol{b} \in \mathcal{R}^n_+$, $\boldsymbol{u} \in \mathcal{R}^n$ the following inequality holds:

$$(1/\max_i\{b_i\}) \cdot ||\boldsymbol{u}|| \leq ||\boldsymbol{u}||_{\boldsymbol{b}}, \qquad (28)$$

and therefore if $||\boldsymbol{u}||_{\boldsymbol{b}_j} < r_{kj}$ then $||\boldsymbol{u}|| < (\max_i\{b_{ji}\}) \cdot r_{kj}$. In the sequel we set $sr_{kj} = \max_i\{b_{ji}\} \cdot r_{kj}$ for all k, j. On the basis of this notation and inequality (28), inequalities (27) imply

$$||\boldsymbol{x} - \boldsymbol{a}_{\pi(k)}|| < sr_{k,\pi(k)} \quad \text{for} \quad k = 1,\ldots,l. \qquad (29)$$

Summing the above inequalities we obtain

$$\sum_k ||\boldsymbol{x} - \boldsymbol{a}_{\pi(k)}|| < \sum_k sr_{k,\pi(k)}. \qquad (30)$$

Now, reversing the implication and employing the properties of norms in $\mathcal{R}^n$ we get the following theorem.

Theorem 1: Let $\Pi_{m,l}$ be the set of all permutations with repetition of length l from the set of rule indices $\{1,\ldots,m\}$. Let $\overline{\boldsymbol{a}}_\pi$ be the average vector formed from vectors $\boldsymbol{a}_{\pi(1)},\ldots,\boldsymbol{a}_{\pi(l)}$. If for every $\pi \in \Pi_{m,l}$

$$\frac{1}{2}\sum_k ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(k)}|| \geq \sum_k sr_{k,\pi(k)}, \qquad (31)$$

then the system is coherent.

Proof: The theorem is a direct corrolarly of inequality (30). By the triangle inequality, for any $\boldsymbol{x}, \boldsymbol{a}_{\pi(k)} \in \mathcal{R}^n$ we have

$$\sum_k ||\boldsymbol{x} - \boldsymbol{a}_{\pi(k)}|| \geq ||\sum_k (\boldsymbol{x} - \boldsymbol{a}_{\pi(k)})|| = ||l\boldsymbol{x} - \sum_k \boldsymbol{a}_{\pi(k)}|| = l \cdot ||\boldsymbol{x} - \overline{\boldsymbol{a}}_\pi||. \qquad (32)$$

We have also the following l inequalities valid ($\pi(1) = j_1$, $\pi(2) = j_2$, ...):

$$||\boldsymbol{x} - \boldsymbol{a}_{\pi(1)}|| + ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{x}|| \geq ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(1)}||, \qquad (33)$$

$$||\boldsymbol{x} - \boldsymbol{a}_{\pi(2)}|| + ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{x}|| \geq ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(2)}||, \qquad (34)$$

$$\vdots$$

$$||\boldsymbol{x} - \boldsymbol{a}_{\pi(l)}|| + ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{x}|| \geq ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(l)}||. \qquad (35)$$

Summing these inequalities we get

$$\sum_k ||\boldsymbol{x}-\boldsymbol{a}_{\pi(k)}|| + l\cdot||\overline{\boldsymbol{a}}_\pi - \boldsymbol{x}|| \geq \sum_k ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(k)}||. \quad (36)$$

Since $l\cdot||\overline{\boldsymbol{a}}_\pi - \boldsymbol{x}|| = l\cdot||\boldsymbol{x}-\overline{\boldsymbol{a}}_\pi||$ and inequality (32) holds the above gives

$$\sum_k ||\boldsymbol{x}-\boldsymbol{a}_{\pi(k)}|| + \sum_k ||\boldsymbol{x}-\boldsymbol{a}_{\pi(k)}|| \geq \sum_k ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(k)}||, \quad (37)$$

$$\sum_k ||\boldsymbol{x}-\boldsymbol{a}_{\pi(k)}|| \geq \frac{1}{2}\sum_k ||\overline{\boldsymbol{a}}_\pi - \boldsymbol{a}_{\pi(k)}|| \quad (38)$$

for every $\boldsymbol{x}\in\mathcal{R}^n$. Therefore the minimum of the left side is bounded from below by the constant which is given by the right side. If this constant is greater or equal to the sum of $sr_{k,\pi(k)}$ then there cannot exists an $\boldsymbol{x}$ for which inequality (30) holds and intersection I_π for this π is empty. If inequality (31) holds for every $\pi\in\Pi_{m,l}$ then the system is coherent. □

Theorem 1 states the sufficient condition for checking the emptiness of intersection I_π for given π. The problem is that in order to test the coherence of a radial I-FS with NCs we have to perform generally m^l tests for all permutations π from $\Pi_{m,l}$ set. This number can be slightly lowered on the basis of the following lemma.

Lemma 2: If a radial I-FS with NCs is coherent, then for each rule j there must exist an action y_k such that $\mu_{kj}=1$.

Proof: Due to the properties of *act* function we have $r_{kj}=0$ iff $\mu_{kj}=1$. If for some rule j and all actions y_k would be $\mu_{kj}<1$, then also $r_{kj}>0$ for all k and $\boldsymbol{a}_j\in RIC_{kj}$ for all k. Considering permutation $\pi=(j,\ldots,j)$ we would get $\boldsymbol{a}_j\in I_\pi$ and the system would be incoherent. Let us note that if $r_{kj}=0$, i.e., if $\mu_{kj}=1$, then $RIC_{kj}=\emptyset$ and $\boldsymbol{a}_j\notin RIC_{kj}$. □

The direct corollarly of the above lemma is the fact that if the necessary condition is satisfied, which is our case, see Definition 1, then the number of permutations which have to be tested can be lowered to $(m-1)^l$ and only proper permutations have to be generated for testing. A permutation is proper if there exists at least two k_1, k_2 such that $\pi(k_1)\neq\pi(k_2)$.

We proceed by introducing Table 2 which is similar to Table 1 and contains in each cell the value of sr_{kj}.

As $r_{kj}\in[0,+\infty]$ and $\max_i\{b_{ji}\}>0$, we have $sr_{kj}\in[0,+\infty]$. $sr_{kj}=0$ iff $r_{kj}=0$, which corresponds to $\mu_{kj}=1$ and consequently to $RIC_{kj}=\emptyset$. So if there is zero in the kth row and the jth column of Table 2, then $I_\pi=\emptyset$ for permutations having $\pi(k)=j$ and these permutations need not be tested.

On the other hand, $sr_{kj}=+\infty$ iff $r_{kj}=+\infty$. This corresponds to the situation of $\mu_{kj}=0$ and *act* being of (1)(b) type. In this case the region of incoherence is given by whole space $\mathcal{R}^n$. Actually, for any input $\boldsymbol{x}$ to the system we have $A_j(\boldsymbol{x})>0$ and therefore y_k can never occur in the output of the system.

Concerning the last row of Table 2, we denote by $k^*(j)$ the index k for which maximum of sr_{kj} is reached in the jth column of the table. Thus $k^*(j)=\text{argmax}_k\{sr_{kj}\}$ and $rs_{\max,j}=rs_{k^*(j),j}$.

Table 2
Incoherence limit points

	$j=1$	$j=2$	…	$j=m$
$k=1$	sr_{11}	sr_{12}	…	sr_{1m}
$k=2$	sr_{21}	sr_{22}	…	sr_{2m}
			⋮	
$k=l$	sr_{l1}	sr_{l2}	…	sr_{lm}
max	$sr_{\max,1}$	$sr_{\max,2}$	…	$sr_{\max,m}$

Now, let the so-called *symmetric regions of incoherence* $SRIC_{kj}$ for given k,j be specified according to the following formula:

$$SRIC_{kj}=\{\boldsymbol{x}\in\mathcal{R}^n \mid ||\boldsymbol{x}-\boldsymbol{a}_j||<sr_{kj}\}. \quad (39)$$

Recalling the discussion presented when sr values were introduced (formula (28)) we can see that if $\boldsymbol{x}\in RIC_{kj}$, then $\boldsymbol{x}\in SRIC_{kj}$, i.e., $RIC_{kj}\subseteq SRIC_{kj}$ for all k,j and also $RIC_{kj}\subseteq SRIC_{k^*(j),j}$ for constant j and $k=1,\ldots,l$. The specification of $SRIC_{k^*(j),j}$ enables us to state the following theorem.

Theorem 2: Let a radial I-FS with NCs consists of m rules. Let for any pair of different rules $j_1,j_2\in\{1,\ldots,m\}$, $j_1\neq j_2$, the following holds:

$$||\boldsymbol{a}_{j1}-\boldsymbol{a}_{j2}|| \geq sr_{\max,j_1}+sr_{\max,j_2}. \quad (40)$$

Then the system is coherent.

Proof: To start let us show that for the special case of $l=2$, the factor 0.5 can be ommited in formula (31). Let $SRIC_i=\{\boldsymbol{x}\in\mathcal{R}^n \,|\, ||\boldsymbol{x}-\boldsymbol{a}_i||<sr_i\}$ for some $\boldsymbol{a}_i\in\mathcal{R}^n$, $sr_i\geq 0$, $i=\{1,2\}$. Let $SRIC_1\cap SRIC_2\neq\emptyset$, then there exists an $\boldsymbol{x}$ such that $||\boldsymbol{x}-\boldsymbol{a}_1||<sr_1$, $||\boldsymbol{x}-\boldsymbol{a}_2||<sr_2$ and also $||\boldsymbol{x}-\boldsymbol{a}_1||+||\boldsymbol{x}-\boldsymbol{a}_2||<sr_1+sr_2$. Since by the triangle inequality we have $||\boldsymbol{x}-\boldsymbol{a}_1||+||\boldsymbol{x}-\boldsymbol{a}_2||\geq||\boldsymbol{a}_1-\boldsymbol{a}_2||$, the minimum of the left side is reached for both $\boldsymbol{x}=\boldsymbol{a}_1$, $\boldsymbol{x}=\boldsymbol{a}_2$ and has the value $||\boldsymbol{a}_1-\boldsymbol{a}_2||$. So we can conclude that if the intersection of two hyperballs $SCRI_i$, $i\in\{1,2\}$ is non-empty then $||\boldsymbol{a}_1-\boldsymbol{a}_2||<sr_1+sr_2$.

Now, assume that under the validity of the above theorem the system is incoherent. Then there must exist a proper permutation π such that I_π is non-empty. Let $j_1=\pi(k_1)\neq\pi(k_2)=j_2$ for some k_1, k_2, then $RIC_{k_1,j_1}\cap RIC_{k_2,j_2}\neq\emptyset$. As $RIC_{k_i,j_i}\subseteq SRIC_{k^*(j_i),j_i}$ for $i=\{1,2\}$ then we have also $SRIC_{k^*(j_1),j_1}\cap SRIC_{k^*(j_2),j_2}\neq\emptyset$ which implies $||\boldsymbol{a}_{j1}-\boldsymbol{a}_{j2}||<sr_{\max,j_1}+sr_{\max,j_2}$. A contradiction. □

The above theorem reduces the number of inequalities that have to be tested in order to check the coherence of the system to $m(m-1)/2$, as inequalities (40) are symmetric with respect to j_1, j_2. However, the reduction of the number of tests is for the price of lowering the specificity of the tests. That is, the testing according to Theorem 2 will state more systems possibly incoherent than when the testing according to Theorem 1 is adopted. The reason for this fact is that in the case of Theorem 1 we check the emptiness of intersection of l hyperballs. In the case of Theorem 2, regardless the number l of actions is, we always test the intersection of only two hyperballs.

4. Conclusions

In the paper we have introduced the concept of the radial implicative fuzzy system with nominal consequents (radial I-FS with NCs). We have presented its computational model and investigated the notion of coherence for this class of fuzzy systems. We have presented two theorems stating two sufficient conditions (in fact set of conditions/inequalities) which assure the coherence of a radial I-FS with NCs.

The first sufficient condition, stated by Theorem 1, is based on the radial property which helps to investigate the coherence, however, it suffers from the curse of dimensionality because the number of tests to verify the coherence is generally $(m-1)^l$, where m is the number of rules and l is the number of actions.

The second sufficient condition, stated by Theorem 2, needs only $m(m-1)/2$ tests for the verification of coherence. However, the specificity of the second sufficient condition is lower than of the first condition. That is why, we recommend to use the tests according to the first condition anywhere where this is computationally tractable (low values of m and mainly l).

Because the lack of specificity during the tests according to the second sufficient condition, the next direction in our research is to elaborate an efficient tree-like algorithm for testing the coherence based on the first sufficient condition. The basic idea of this algorithm is not to test permutations which are clear to yield empty intersections because they contain empty sub-intersections.

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