Paper

Mesh-free approach to Helmholtz equation based on radial basis functions

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Abstract—Recently, a radial basis functions (RBFs) method, which was originally proposed for interpolation problems, has been developed and applied to solve partial differential equations and eigenproblems. Properties of that method (mesh-free algorithm) allows one to use it in many areas, including electromagnetics. In this paper the mesh-free RBF method for solving Helmholtz equation was applied and a new adaptive algorithm for defining the set of interpolation centers was proposed. Using the proposed approach the cutoff wavelengths and the field distribution in cylindrical waveguides of arbitrary cross-section were calculated with a high accuracy.

Keywords—Helmholtz equation, radial basis functions, mesh-free method, cylindrical waveguides.

1. Introduction

Radial basis functions (RBFs) were originally proposed for a multidimensional interpolation problem of scattered data [1]. The RBFs method is similar to the most of the interpolation methods and involves the same general idea: for a given set of distinct points $\{\mathbf{x}_j\}_{j=1}^N \in \Omega \subset \mathbb{R}^d$ (called interpolation points) the corresponding values are known $\{\Psi(\mathbf{x}_j)\}_{j=1}^N$ and the interpolant $\Psi(\mathbf{x})$ (where $\mathbf{x} \in \Omega$) is chosen such that the interpolation conditions are fulfilled. The expression for the interpolant has a following form:

$$\Psi(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i \phi(||\mathbf{x} - \mathbf{y}_i||), \tag{1}$$

where $||\cdot||$ is the Euclidean norm, the set $\{\mathbf{y}_i\}_{i=1}^N \in \mathbb{R}^d$ is a set of interpolation centers (usually, the same set of points is used for centers and interpolation) and finally $\phi(\cdot)$ is the radial basis function. It has to be noted that $\phi(\cdot)$ is one-argument function $\phi : \mathbb{R}_+ \to \mathbb{R}$.

There is many different types of RBFs. However, as mentioned before, choosing a particular form of RBF depends on the type of the problem. A few of the most commonly used RBFs include: the Gaussian function $\phi(r) = \mathrm{e}^{-r^2}$, the multiquadric function $\phi(r) = \sqrt{1+r^2}$ or the Wendland function which will be described later [2–4].

The argument r of the function $\phi(\cdot)$ is usually scaled by the factor c: $r \to \frac{r}{c}$, where $c \in \mathbb{R}_+$ is called a shape parameter. If c is properly selected, the accuracy of the method increases, but despite intensive research [4] the choice of the parameter c remains an unsolved problem.

For a given form of RBF, the value of the parameter c and centers distribution $\{\mathbf{y}_i\}$, one gets the interpolant from substitution of given data to Eq. (1):

$$\Psi(\mathbf{x}_j) = \sum_{i=1}^{N} \alpha_i \phi(||\mathbf{x}_j - \mathbf{y}_i||), \qquad j = 1, ..., N. \tag{2}$$

Using a matrix notation $\mathcal{A}\boldsymbol{\alpha} = \boldsymbol{\Psi}$, where $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_N]^T$, $\boldsymbol{\Psi} = [\Psi(\mathbf{x}_1), \cdots, \Psi(\mathbf{x}_N)]^T$ and the elements of the matrix $\mathcal{A}$ have a form $[\mathcal{A}]_{i,j} = \phi(||\mathbf{x}_i - \mathbf{y}_j||)$ for $i, j = 1, \ldots, N$.

Recently RBFs have been also applied to solve partial differential equations [5] and eigenproblems [3]. The standard numerical methods such as the finite element method or the finite-difference method require mesh generation, which often is very burdensome task, especially when the contour of the domain is of complex geometry. The RBFs method is based on extrapolation of scattered data, therefore it is very suitable schemes for problems defined in irregular geometries; its algorithm is totally grid-free.

In this paper we further develop the technique proposed in [6] to find the cutoff wavelengths and the field distribution in cylindrical waveguides of arbitrary cross-section shapes. Instead of defining the set of centers *a priori*, we propose a new adaptive algorithm.

2. Formulation

Let us assume that the wave propagates in the z-direction inside a homogeneous and uniform cylinder of arbitrary but homogeneous cross-section Ω. The problem is governed by the scalar Helmholtz equation:

$$\nabla^2 \Psi(\mathbf{x}) + k^2 \Psi(\mathbf{x}) = 0, \tag{3}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and k is the cutoff wavenumber. For TM modes $H_z = 0$, $E_z = \Psi(\mathbf{x})$ and function $\Psi(\mathbf{x})$ satisfies the Dirichlet conditions imposed on boundary $\partial\Omega$, whereas for TE modes $E_z = 0$, $H_z = \Psi(\mathbf{x})$ and $\Psi(\mathbf{x})$ satisfies the Neumann conditions.

The problem can be solved by applying a modified form of the interpolant [6]:

$$\Psi(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \phi(||\mathbf{x} - \mathbf{y}_i||) + \sum_{i=n+1}^{N} \alpha_i (\mathbf{x} - \mathbf{y}_i) \cdot \nabla \phi(||\mathbf{x} - \mathbf{y}_i||), \tag{4}$$

where the set of centers $\{\mathbf{y}_i\}_{i=1}^N \in \bar{\Omega}$ ($\bar{\Omega}$ consists of Ω and boundary $\partial\Omega$) are chosen arbitrarily, but in such a way

that n of y_i are inside Ω and the rest of them are on boundary $\partial\Omega$. The second term in Eq. (4) is due to the first-order approximation by the Taylor series expansion and it is included to improve the accuracy of the derivatives. Function $\phi(\cdot)$ is the Wendland radial basis function of the following form

$$\phi(r) = (1-r)_+^8 (32r^3 + 25r^2 + 8r + 1), \tag{5}$$

where $(1-r)_+ = \begin{cases} 1-r, & r \in (0,1) \\ 0, & otherwise \end{cases}$.

Substituting Eq. (4) into Eq. (3) one gets

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i \nabla^2 \phi(||\mathbf{x}-\mathbf{y}_i||) \\ &+ \sum_{i=n+1}^{N} \alpha_i \nabla^2 \left[(\mathbf{x}-\mathbf{y}_i)\cdot\nabla\phi(||\mathbf{x}-\mathbf{y}_i||)\right] \\ &= -k^2 \left[\sum_{i=1}^{n} \alpha_i \phi(||\mathbf{x}-\mathbf{y}_i||) \right. \\ &\left. + \sum_{i=n+1}^{N} \alpha_i (\mathbf{x}-\mathbf{y}_i)\cdot\nabla\phi(||\mathbf{x}-\mathbf{y}_i||)\right]. \end{aligned} \tag{6}$$

Taking the set of interpolation points identical to the set of centers and substituting the inner points $\{\mathbf{x}_j\}_{j=1}^{n} \in \Omega$ to Eq. (6) one gets the system of equations which in a matrix notation have a form

$$\mathcal{L}_I \boldsymbol{\alpha}_I + \mathcal{L}_B \boldsymbol{\alpha}_B = -k^2 (\mathcal{A}_I \boldsymbol{\alpha}_I + \mathcal{A}_B \boldsymbol{\alpha}_B). \tag{7}$$

Analogously, for $\{\mathbf{x}_j\}_{j=n+1}^{N} \in \partial\Omega$ substituted into Dirichlet or Neumann conditions one gets

$$\mathcal{B}_I \boldsymbol{\alpha}_I + \mathcal{B}_B \boldsymbol{\alpha}_B = 0, \tag{8}$$

where $\mathcal{B}_I$ and $\mathcal{B}_B$ have a different form for TM modes or TE modes. Eliminating $\boldsymbol{\alpha}_B$ in Eq. (7) using Eq. (8) one gets the following (n-dimensional) generalized eigenproblem

$$\begin{aligned} &\left[\mathcal{L}_I - \mathcal{L}_B\left(\mathcal{B}_B^{-1}\mathcal{B}_I\right)\right]\boldsymbol{\alpha}_I \\ &= -k^2\left[\mathcal{A}_I - \mathcal{A}_B\left(\mathcal{B}_B^{-1}\mathcal{B}_I\right)\right]\boldsymbol{\alpha}_I. \end{aligned} \tag{9}$$

The field distribution represented by Eq. (4) can be obtained from eigenvectors $\boldsymbol{\alpha}_I$ (and $\boldsymbol{\alpha}_B = -\mathcal{B}_B^{-1}\mathcal{B}_I\boldsymbol{\alpha}_I$) and the cutoff wavenumbers from the eigenvalues of Eq. (9), $\lambda = \dfrac{2\pi}{k}$.

3. Self-adaptive algorithm for choosing the set of interpolation centers

The main aim of that method is to generate the set of interpolation points which for a small number of points gives the highest accuracy of the solution.

In the first step one has to solve eigenproblem for the initial set of points – obtaining the initial eigenvalue and the corresponding initial eigenvector. It is equivalent to obtaining initial function $\Psi(\mathbf{x})$ defined for an an arbitrary point in $\overline{\Omega}$. In the next step one has to take a new arbitrary (usually larger) set of points $\{\widetilde{\mathbf{x}}_m\}_{m=1}^{M} \in \overline{\Omega}$ different than $\{\mathbf{x}_j\}$. For any point $\widetilde{\mathbf{x}}_m$ inaccuracy of the initial solution can be checked – if $\widetilde{\mathbf{x}}_m \in \Omega$, then the interpolation error can be obtained from substituting it into Eq. (3)

$$E_I(\widetilde{\mathbf{x}}_m) = \nabla^2\Psi(\widetilde{\mathbf{x}}_m) + k^2\Psi(\widetilde{\mathbf{x}}_m), \tag{10}$$

when $\widetilde{\mathbf{x}}_m \in \partial\Omega$ the error can be expressed by

$$E_B(\widetilde{\mathbf{x}}_m) = \Psi(\widetilde{\mathbf{x}}_m) \tag{11}$$

for TM modes or

$$E_B(\widetilde{\mathbf{x}}_k) = \mathbf{n}\cdot\nabla\Psi(\widetilde{\mathbf{x}}_k) \tag{12}$$

for TE modes. Using a matrix notation

$$\mathbf{E}_I = \left[\widetilde{\mathcal{L}}_I - \widetilde{\mathcal{L}}_B\mathcal{B}_B^{-1}\mathcal{B}_I + k^2\left(\widetilde{\mathcal{A}}_I - \widetilde{\mathcal{A}}_B\mathcal{B}_B^{-1}\mathcal{B}_I\right)\right]\boldsymbol{\alpha}_I \tag{13}$$

and

$$\mathbf{E}_B = \left[\widetilde{\mathcal{B}}_I - \widetilde{\mathcal{B}}_B\mathcal{B}_B^{-1}\mathcal{B}_I\right]\boldsymbol{\alpha}_I. \tag{14}$$

The values of elements of the vectors $\mathbf{E}_I$ and $\mathbf{E}_B$ correspond to inaccuracy of the initial solution at given points. Selecting elements whose values exceed the assumed inaccuracy one gets a subset of points $\widetilde{\mathbf{x}}_m$ which should be added to the initial set.

This algorithm is computationally by far more efficient than solving the eigenproblem for a set of points consisting of sets: $\{\mathbf{x}_j\}$ and whole $\{\widetilde{\mathbf{x}}_m\}$. It has to be noted that the operation can be repeated until the assumed accuracy is achieved.

4. Numerical results

Numerical tests were made for a few chosen shapes of waveguide cross-sections for which analytical results are known:

- circular waveguide of the radius $R = 1$;
- elliptical waveguide of the eccentricity $e = 0.9$ and the semimajor axis $a = 1$;
- rectangular waveguide of the width $a = 2$ and the height $b = 1$.

As it was noted, the results depend on the distribution of points (the set of centers) as well as shape parameter c. In the results presented, the initial set of the centers has a regular distribution (Fig. 1) and shape parameter $c = 3$.

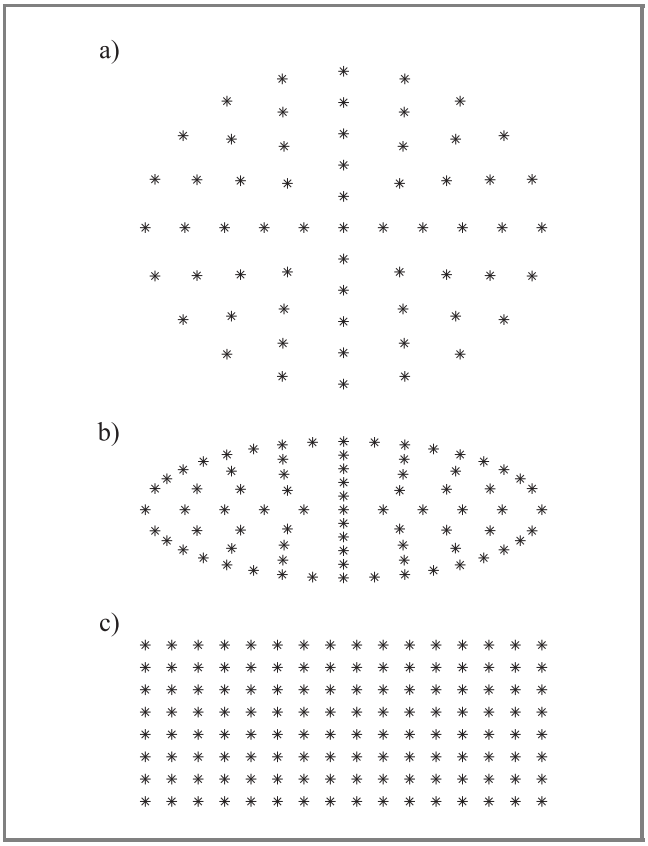

Fig. 1. The initial sets of points for: (a) circular ($N = 61$, $n = 41$); (b) elliptical ($N = 61$, $n = 41$); (c) rectangular ($N = 128$, $n = 84$) waveguides.

After using the self-adaptive algorithm, the density of points increased in regions where the accuracy was too low (Fig. 2).

Tables 1–3 shows the cutoff wavelengths for a few low order TM modes for considered shapes obtained from the standard RBFs method and the RBFs method with a self-adaptive algorithm. Analytical solutions are collected in the last column.

Table 1

The cutoff wavelengths for three lowest order TM modes for circular waveguide

RBF method	RBF with S-A	Analytical
2.6129	2.6127	2.6127
1.6420	1.6400	1.6397
1.2284	1.2238	1.2234

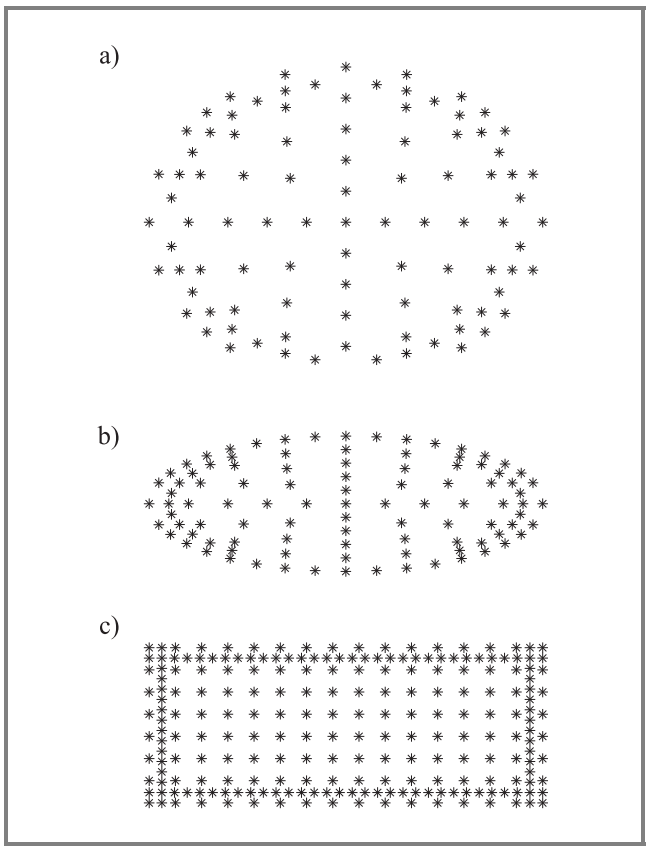

Fig. 2. The enriched (self-adapted algorithm) sets of points for: (a) circular ($N = 113$, $n = 73$); (b) elliptical ($N = 99$, $n = 63$); (c) rectangular ($N = 220$, $n = 168$) waveguides.

It can be seen that more nodes are needed for domains with corners to achieve the required accuracy.

Table 2

The cutoff wavelengths for three lowest order TM modes for elliptical waveguide

RBF method	RBF with S-A	Analytical
1.4912	1.4904	1.4906
1.1619	1.1605	1.1607
0.9420	0.9378	0.9375

Table 3

The cutoff wavelengths for three lowest order TM modes for rectangular waveguide

RBF method	RBF with S-A	Analytical
1.7896	1.7889	1.7889
1.4167	1.4143	1.4142
1.1085	1.1096	1.1094

5. Conclusions

The numerical tests shown in this paper are focused on TM modes analysis, however, it has to be noted that TE modes as well as TM can be found. The proposed method is applicable not only to waveguides of arbitrary cross-section, but also for any 3D structures. However, for regions of complex geometry (especially with re-entrant corners) number of interpolation points can be very large and method may be computationally not efficient (because of matrices which occur in eigenproblem are dense).
One of the essential problems of RBFs method is also selecting a value of a shape parameter c. The parameter c is of great importance for convergence and accuracy of the obtained solutions. Inappropriate choice of c can reduce accuracy or cause ill-conditioning of matrices.

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