Paper

Constant co-polarized echo curves on the Poincare sphere

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Abstract — A set of useful formulae has been presented allowing for computation of the equipower curves on the Poincare sphere of the co-polarized radar returns for both mono- and bi-static scatterings.

Keywords — radar polarimetry, Poincare sphere module of scattering matrix, Kennaugh matrix, Cassini ovals.

1. Introduction

In the English language literature there is a lack of publications treating that very useful "geometrical" presentation of scattering. Basic concepts can be found in technical reports of Kennaugh [1], in Russian book on "Polarization of Radar Signals" written by D. B. Kanareykin [2], and also in the Polish paper presented by this author [3]. But still attempts can be met of researchers trying to find the most convenient ways to draw on the Poincare sphere the curves of constant co-polarized received power scattered backwards from nondepolarizing targets. It is believed that formulae presented beneath will be of some help also for those intending to solve similar problems, i.e., for cross-polarized returns or bistatic scattering.

The method here applied for solution of the problem is based on studies of equation for co-polarized returned power in a most simple form, that means in the so-called "characteristic coordinate system" (CCS) in which Sinclair and Kennaugh matrices depend on minimum number of parameters: two and three real parameters for Sinclair and Kennaugh matrices, respectively. Such approach does not limit the generality of considerations.

2. The co-polarized radar transmission equation

The Jones unit column matrix of monochromatic plane wave of "polarization and phase" P (indicated by the upper index) may be expressed as a function of three the so-called "analytical" angular parameters: γ, δ, ε:

$$[u]_Y^P = \begin{bmatrix} u_Y \\ u_X \end{bmatrix}^P = \begin{bmatrix} \cos\gamma_Y^P \exp\{-j\delta_Y^P\} \\ \sin\gamma_Y^P \exp\{+j\delta_Y^P\} \end{bmatrix} \exp\{-j\varepsilon_Y^P\}. \quad (1)$$

Elements of the matrix depend on the orthogonal null-phase (ONP) polarization basis denoted here by the subscript Y (the lower index). That basis may correspond to the XYZ Cartesian coordinate system, with the propagation axis "Z", or may denote any other ONP basis obtained by applying to the original one a unitary unimodular transformation. Both P and Y can be treated as three parameter tangential phasors (see [4]) on the Poincare sphere. Kronecker square of the Jones matrices eliminates the "phase" parameter:

$$[u]_Y^P \otimes [u]_Y^{P\,*} = \begin{bmatrix} \cos^2\gamma_Y^P \\ \cos\gamma_Y^P \sin\gamma_Y^P \exp\{-j2\delta_Y^P\} \\ \cos\gamma_Y^P \sin\gamma_Y^P \exp\{+j2\delta_Y^P\} \\ \sin^2\gamma_Y^P \end{bmatrix}. \quad (2)$$

So, the resulting unit Stokes four-vector of the wave:

$$[P]_Y^P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix} \begin{bmatrix} \cos^2\gamma_Y^P \\ \cos\gamma_Y^P \sin\gamma_Y^P \exp\{-j2\delta_Y^P\} \\ \cos\gamma_Y^P \sin\gamma_Y^P \exp\{+j2\delta_Y^P\} \\ \sin^2\gamma_Y^P \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \cos 2\gamma_Y^P \\ \sin 2\gamma_Y^P \cos 2\delta_Y^P \\ \sin 2\gamma_Y^P \sin 2\delta_Y^P \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}_Y^P \quad (3)$$

depends on two, though doubled, parameters 2δ and 2γ which can be interpreted as angular coordinates of the polarization point P on the Poincare sphere. The q, u, v coordinates are the three "normed" Stokes parameters of the point P on the sphere.

Consider, e.g., a simplified model of a rain-drop of the shape of an oblate spheroid with vertical axis of symmetry. Its Sinclair matrix for backscattering can be presented as follows:

$$[A]_Y = \left\{ \begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix} e^{j\mu} \right\}_Y; \quad A_2 \geq A_1 \quad (4)$$

with real positive A_2, A_1, and μ. Diagonal elements may denote, e.g., horizontal and vertical polarizabilities with space attenuation factor included. Incidentally, the form of the above matrix is exactly like in the characteristic ONP polarization basis, corresponding to the CCS in the Stokes parameters space. Any symmetrical Sinclair matrix can be transformed to that form by proper change of the basis.

Using the Kronecker square of that matrix

$$[A]_Y \otimes [A]_Y{}^* = \begin{bmatrix} A_2^2 & 0 & 0 & 0 \\ 0 & A_2A_1 & 0 & 0 \\ 0 & 0 & A_1A_2 & 0 \\ 0 & 0 & 0 & A_1^2 \end{bmatrix}_Y \quad (5)$$

the corresponding Kennaugh matrix can be found as follows:

$$[K]_Y = \frac{1}{2}\begin{bmatrix}1&0&0&1\\1&0&0&-1\\0&1&1&0\\0&-j&j&0\end{bmatrix}\begin{bmatrix}A_2^2&0&0&0\\0&A_2A_1&0&0\\0&0&A_1A_2&0\\0&0&0&A_1^2\end{bmatrix}_Y\begin{bmatrix}1&1&0&0\\0&0&1&-j\\0&0&1&j\\1&-1&0&0\end{bmatrix} = $$

$$= \begin{bmatrix}a_1&b_1&0&0\\b_1&a_1&0&0\\0&0&a_3&0\\0&0&0&-a_3\end{bmatrix}_Y \tag{6}$$

with

$$a_1 = \frac{A_2^2 + A_1^2}{2}, \quad b_1 = \frac{A_2^2 - A_1^2}{2}, \quad a_3 = A_2A_1; \quad a_2^2 - a_3^2 = b_1^2$$

(To simplify formulae, in what follows, indices of matrices will be omitted).

For transmit/receive “effective” Stokes four-vectors of complete polarization, corresponding to the unit total intensity, the co-polarized received power is

$$P_c(\mathrm{P}) = P_c(\mathrm{q, u, v}) =$$

$$= \frac{1}{2}[1\ \mathrm{q\ u\ v}]\begin{bmatrix}a_1&b_1&0&0\\b_1&a_1&0&0\\0&0&a_3&0\\0&0&0&-a_3\end{bmatrix}\begin{bmatrix}1\\\mathrm{q}\\\mathrm{u}\\\mathrm{v}\end{bmatrix} =$$

$$= \frac{1}{2}\left\{a_1\left(1+\mathrm{q}^2\right) + 2b_1\mathrm{q} + a_3\left(\mathrm{u}^2 - \mathrm{v}^2\right)\right\} \tag{7}$$

with

$$\mathrm{q}^2 + \mathrm{u}^2 + \mathrm{v}^2 = 1\,. \tag{8}$$

The last equation presents the Poincare sphere of unit radius in the q, u, v coordinates. For P_c(P) = const., Eq. (7) determines another surface. Its cross-section with the sphere (8) traces curves of constant P_c powers on that sphere.

3. The Poincare sphere model of the backscattering matrix

The Poincare sphere model of a Sinclair and/or Kennaugh scattering matrix will be constructed with the following parameters:

$$r = \frac{A_2 + A_1}{2}, \quad e = \frac{A_2 - A_1}{2}, \quad d = \sqrt{A_2A_1}\,. \tag{9}$$

Here r means the radius of the sphere in coordinates

$$x = r\mathrm{q}\,, \quad y = r\mathrm{u}\,, \quad z = r\mathrm{v} \tag{10}$$

what results in the equation of the sphere model:

$$x^2 + y^2 + z^2 = r^2\,. \tag{11}$$

The Poincare sphere model can show the mechanism of scattering geometrically as follows. Any point P of incident polarization, after inversion through the point I defined by coordinates: $x = -e$, $y = z = 0$, and after rotation about the z axis by 180°, determines the scattered polarization point S. The scattered power is equal to the square of the IP distance. However, to obtain the power received by the same transmit/receive antenna, the scattered power should be multiplied by the square of cosine of one half of an angle between the S and P points:

$$P_c(\mathrm{P}) = (\mathrm{IP})^2\cos^2\left(\angle\frac{\mathrm{SP}}{2}\right). \tag{12}$$

4. Construction of constant received power curves by the use of Cassini ovals

Another useful equality has been first derived by Kennaugh. It allows for expressing the co-polarization received power by a product of squares of two distances between P point and the so-called CO-POL NULL points O_1 and O_2 of coordinates: $x = -e$, $y = 0$, $z = \pm d$:

$$P_c(\mathrm{P}) = \frac{(\mathrm{O_1P})^2 \times (\mathrm{O_2P})^2}{(2r)^2} = \frac{c^4}{(A_2 + A_1)^2}\,. \tag{13}$$

with

$$c^2 = (\mathrm{O_1P}) \times (\mathrm{O_2P}) = (A_2 + A_1)\sqrt{P_c(\mathrm{P})}\,. \tag{14}$$

Considering P points not necessarily located on the Poincare sphere surface, we obtain for c^2 = const. a surface of rotational symmetry, with the axial cross-section being a Cassini oval with focuses in points O_1 and O_2. That surface is given by the known equation of the Cassini oval:

$$\left.\begin{aligned}&\left[\rho^2 + z^2\right]^2 + 2d^2\left[\rho^2 - z^2\right] = c^4 - d^4\\&\rho^2 = (x+e)^2 + y^2\end{aligned}\right\} \tag{15}$$

for which the following coordinates of its axis and focuses have been determined:

$$y_{axis} = 0, \quad x_{axis} = -e, \quad z(x_{axis}, y_{axis}|P_c = 0) = \pm d\,. \tag{16}$$

Comparing the last expressions with Eq. (15) we see that for $P_c = 0$, resulting in $c = 0$, the surface reduces to two points only, the focuses of the Cassini oval.

5. Cross-section of the Poincare sphere model with parabolic and hyperbolic cylinders

Simpler presentation of the constant received power curves can be found eliminating one of Stokes parameters, v or u, from Eq. (7). That way projections of the P_c(P) = const. curves from the Poincare sphere of unit radius on to the qu and qv planes can be found.

By substituting $\mathrm{v}^2 = 1 - \mathrm{q}^2 - \mathrm{u}^2$ in Eq. (7) we obtain a set of ellipses:

$$P_c(\mathrm{P}) = (e + r\mathrm{q})^2 + d^2\mathrm{u}^2 = \frac{c^4}{4r^2} = \mathrm{const.} \tag{17}$$

or, for the model of the radius r, in the xyz coordinate system:

$$\frac{c^4}{4} = r^2\Big((e + r\mathrm{q})^2 + d^2\mathrm{u}^2\Big) = r^2\Big(e + x\Big)^2 + d^2y^2\,. \tag{18}$$

The semi-axes of the new larger ellipses can be found as follows:

$$\left.\begin{aligned} &\frac{4r^2(e+x)^2}{c^4} + \frac{4d^2y^2}{c^4} = 1 = \frac{(e+x)^2}{b^2} + \frac{y^2}{a^2} \\ &a = \frac{c^2}{2d}\,, \quad b = \frac{c^2}{2r}\,; \quad a \geq b\,. \end{aligned}\right\}. \tag{19}$$

Similarly, by substituting $\mathrm{u}^2 = 1 - \mathrm{q}^2 - \mathrm{v}^2$ in Eq. (7) we obtain a set of hyperbolae:

$$P_c(\mathrm{P}) = (r + e\mathrm{q})^2 - d^2\mathrm{v}^2 = \frac{c^4}{4r^2} = \mathrm{const.} \tag{20}$$

and, for the model of the radius r, in the xyz coordinate system:

$$\frac{c^4}{4} = r^2\Big((r + e\mathrm{q})^2 - d^2\mathrm{v}^2\Big) = \Big(ex + r^2\Big)^2 - d^2z^2\,. \tag{21}$$

Parameters a and b of the new larger hyperbolae can be found also similarly:

$$\left.\begin{aligned} &\frac{4e^2\left(x + \frac{r^2}{e}\right)^2}{c^4} - \frac{4d^2z^2}{c^4} = 1 = \frac{\left(x + \frac{r^2}{e}\right)^2}{a^2} - \frac{z^2}{b^2} \\ &a = \frac{c^2}{2e}\,, \quad b = \frac{c^2}{2d} \end{aligned}\right\}. \tag{22}$$

It is worth noting that asymptotes of hyperbolae are tangent to the sphere in O_1 and O_2 points. They are crossing at point I' with coordinate $x = -(r^2/e)$ what means that I and I' points are mutual reflections in the sphere surface.
It can be checked easily (by substituting Eqs. (21) or (18) to (15) and eliminating in Eq. (15) y or z, respectively) that projections of constant received power curves obtained by the two methods agree precisely.

6. Other useful formulae

The parameter of constant level of the received signal versus $(P_c)_{\max} = A_2^2$ can be computed from the formulae

$$L = P_c\ [\mathrm{dB}] = 10\log\frac{(P_c)_{\max}}{P_c} = 20\log\frac{c^2_{\max}}{c^2} > 0\,, \tag{23}$$

where

$$c^2 = (A_2{+}A_1)\sqrt{P_c}, \quad c^2_{\max} = 2r(r{+}e) = (A_2{+}A_1)A_2\,. \tag{24}$$

Hyperbolae and ellipses for given L are the following functions of x:

$$z(L,x) = \pm\frac{\sqrt{(ex{+}r^2)^2 - \frac{c^4}{4}}}{d}\,, \quad y(L,x) = \pm\frac{\sqrt{\frac{c^4}{4} - r^2(e{+}x)^2}}{d}\,, \tag{25}$$

with

$$\frac{c^4}{4} = \left(\frac{r(r+e)}{\log^{-1}(L/20)}\right)^2. \tag{26}$$

Ellipses can be also presented in cylindrical coordinates by a formula

$$\rho = \frac{c^2}{2\sqrt{r^2 - e^2\sin^2\varphi}}\,, \tag{27}$$

where the φ angle is being taken from the xz plane.

7. Conclusions

The problem has been solved for symmetrical Sinclair and Kennaugh matrices. In cases of the bistatic scattering, or matrices corresponding to cross-polarized received powers, we deal with nonsymmetrical matrices. The form of Sinclair matrices in their characteristic ONP polarization bases is antisymmetrical. However, when using the same transmit/receive polarization vectors, the problem remains exactly the same like for the symmetrical matrices. What should be done is to take as the Sinclair scattering matrix its symmetrical part only because the antisymmetrical elements are being eliminated in the transmission equation.

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